Preserving Piece-wise Linearity in Fuzzy Interpolation

Zhiheng Huang and Qiang Shen

Abstract—**Fuzzy interpolative reasoning plays an important role in fuzzy modelling as it not only helps to reduce the number of rules in a rule base, but also provides an inference mechanism for sparse rule bases. In interpolation, it is desirable to preserve piece-wise linearity as piece-wise linear results can thus be inferred from piece-wise linear rules and observations. This safely ensures the ignorance of non-characteristic points in performing interpolations. However, almost all existing fuzzy interpolative reasoning methods do not preserve piecewise linearity for general polygonal fuzzy sets. This paper, based on the work of [1], proposes a new interpolative method which preserves this property.**

I. INTRODUCTION

Fuzzy rule interpolation helps reduce the complexity of fuzzy models and supports inference in systems that employ sparse rule sets [2]. With interpolation, fuzzy rules which may be approximated using their neighboring rules can be omitted from the rule base. This leads to the complexity reduction of fuzzy models. When observations have no overlap with the antecedent values of the rules given, classical fuzzy inference methods have no rule to fire, but interpolative reasoning methods can still obtain certain conclusions. Despite these significant advantages, earlier work in fuzzy interpolative reasoning does not guarantee the convexity of the derived fuzzy sets [3], [4], which is often a crucial requirement of fuzzy reasoning to attain more easily interpretable practical results.

There has been considerable work reported in the literature to eliminate the non-convexity drawback. For instance, Vas, Kalmar and Kóczy have proposed an algorithm [5] that reduces the problem of non-convex conclusions. Qiao, Mizumoto and Yan [6] have published an improved method which uses similarity transfer reasoning to guarantee the attainment of convex results. Hsiao, Chen and Lee [7] have introduced a new interpolative method which exploits the slopes of the fuzzy sets. General fuzzy interpolation and extrapolation techniques [8], and a modified α-cut based method [9], have also been proposed. In addition, Bouchon, Marsala and Rifqi have created an interpolative method by exploiting the concept of graduality [10], and Yam and Kóczy [11], [12] have proposed a fuzzy interpolative method based on Cartesian representation.

Nevertheless, some of the existing methods may not be able to obtain unique as well as normal and convex fuzzy (NCF) results. Others may only apply to simple fuzzy membership functions limited to triangular or trapezoidal. More significantly, none of these approaches preserves the piecewise linearity property. Here, piece-wise linearity means that interpolation can be computed using only characteristic points that describe the given polygonal fuzzy sets. In other words, the interpolation of non-characteristic points which lie

between two characteristic points leads to non-characteristic points which also lie between the two interpolated characteristic points. This paper, based on the initial work as presented in [1], proposes a method which overcomes these problems.

The rest of the paper is organized as follows. Section II briefly introduces the general representative value for arbitrary complex polygonal fuzzy sets. Section III describes the proposed interpolative reasoning method. Section IV gives the proof of preservation of piecewise linearity for the proposed method. Section V presents a comparative study between the work of [1] and the newly proposed method. Finally, Section VI concludes the paper and points out further research.

II. GENERAL REPRESENTATIVE VALUE

The work of [1] follows transformation based approach to performing interpolation. To facilitate such an approach, the *representative value* (RV) of the (polygonal) fuzzy sets involved must be defined first. This value captures important information such as the overall location of a fuzzy set, and will be used as the guide to carry out interpolations. Consider an arbitrary polygonal fuzzy set with n characteristic points, $A = (a_0, \ldots, a_{n-1})$, as shown in Fig. 1. It is assumed that the arbitrary fuzzy sets mentioned in this paper exclude the ones which have more than one peak interval (of a full membership value). Given such an assumption, the fuzzy set
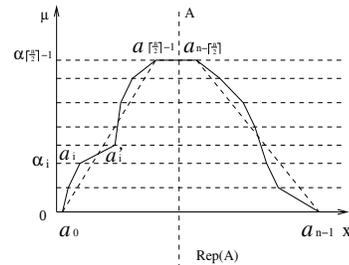

Fig. 1. The RV of an arbitrary polygonal fuzzy set

A has $\lfloor \frac{n}{2} \rfloor$ *supports* (horizontal intervals between every pair of characteristic points which have the same membership value) and $2(\lceil \frac{n}{2} \rceil - 1)$ *slopes* (non-horizontal intervals between every pair of consecutive characteristic points), where $\lfloor \frac{n}{2} \rfloor$ is the smallest integer that is not less than $\frac{n}{2}$, and $\lceil \frac{n}{2} \rceil$ is the largest integer that is not greater than $\frac{n}{2}$. Note that the two top characteristic points (of the membership value 1) do not have to be different. Although this figure explicitly assumes that evenly paired characteristic points are given at each α-cut level, this does not affect the generality of the fuzzy set as artificial characteristic points can be introduced to construct evenly paired characteristic points. Given such an arbitrary polygonal fuzzy set its general RV is defined by

$$Rep(A) = \sum_{i=0}^{n-1} w_i a_i, \qquad (1)$$

Zhiheng Huang is with the EECS Department, University of California at Berkeley, CA 94720, US (phone: +1 510 6434523; email: zhiheng@eecs.berkeley.edu).

Qiang Shen is with the Department of Computer Science at Aberystwyth University, SY23 3DB, Wales, UK (phone: +44 1970 621825; email: qqs@aber.ac.uk).

where w_i is the weight assigned to point a_i.

Specifying the weights is necessary for a given application. The simplest case (which is called the *average RV* hereafter) is that all points take the same weight value, i.e., $w_i = \frac{1}{n}$.

An alternative definition named the *weighted average RV* assumes that the weights increase upwards from the bottom support to the top support, to reflect the significance of different fuzzy membership values. For instance, assuming the weights increase upwards from $\frac{1}{2}$ to 1, such an RV can be defined by

$$Rep(A) = \frac{\sum_{i=0}^{\lceil \frac{n}{2} \rceil - 1} \frac{1+\alpha_i}{2}(a_i + a_{n-1-i})}{\sum_{i=0}^{\lceil \frac{n}{2} \rceil - 1} \frac{1+\alpha_i}{2}}. \tag{2}$$

One of the most widely used defuzzification methods – the center of core can also be used to define the *center of core RV*. In this case, the RV is solely determined by those points with a fuzzy membership value of 1:

$$Rep(A) = \frac{1}{2}(a_{\lceil \frac{n}{2} \rceil - 1} + a_{n - \lceil \frac{n}{2} \rceil}). \tag{3}$$

III. PROPOSED INTERPOLATION

A. Construct the Intermediate Rule

To be concise, the simplest case is herein used to illustrate the underlying techniques for fuzzy interpolation. Given two adjacent rules:

$$If \ X \ is \ A_1 \ then \ Y \ is \ B_1,$$
$$If \ X \ is \ A_2 \ then \ Y \ is \ B_2,$$

which are denoted as $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ respectively, and an observation A^* which is located between fuzzy sets A_1 and A_2, interpolation can be carried out to achieve a fuzzy result B^*. In another form, this simplest case can be represented through the *modus ponens* interpretation (4), and shown in Fig. 2.
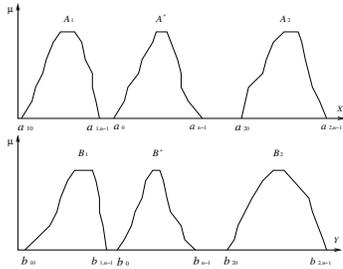

Fig. 2. Interpolation with arbitrarily polygonal fuzzy membership functions

$$\begin{array}{l} \text{observation: } X \text{ is } A^* \\ \text{rules: if } X \text{ is } A_1, \text{ then } Y \text{ is } B_1 \\ \qquad \text{if } X \text{ is } A_2, \text{ then } Y \text{ is } B_2 \\ \hline \text{conclusion: } Y \text{ is } B^*? \end{array} \tag{4}$$

Here, $A_i = (a_{i0}, \ldots, a_{i,n-1})$, $B_i = (b_{i0}, \ldots, b_{i,n-1})$, $i = \{1, 2\}$, and $A^* = (a_0, \ldots, a_{n-1})$, $B^* = (b_0, \ldots, b_{n-1})$.

As with the work of [1], [13], the proposed interpolation method begins with constructing a new fuzzy set A' which has the same RV as A^*. To support this, the distance between A_1 and A_2 is herein defined by

$$d(A_1, A_2) = d(Rep(A_1), Rep(A_2)). \tag{5}$$

An interpolative ratio λ_{Rep} $(0 \leq \lambda_{Rep} \leq 1)$ is introduced to represent the important impact of A_2 upon the construction of A':

$$\lambda_{Rep} = \frac{d(A_1, A^*)}{d(A_1, A_2)} = \frac{d(Rep(A_1), Rep(A^*))}{d(Rep(A_1), Rep(A_2))}. \tag{6}$$

That is to say, if $\lambda_{Rep} = 0$, A_2 plays no part in constructing A', while if $\lambda_{Rep} = 1$, A_2 plays a full role in determining A'. Then, by using the simplest linear interpolation, a'_i, $i = \{0, \ldots, n-1\}$, of A' are calculated as follows:

$$a'_i = (1 - \lambda_{Rep})a_{1i} + \lambda_{Rep}a_{2i}, \tag{7}$$

which are collectively abbreviated to

$$A' = (1 - \lambda_{Rep})A_1 + \lambda_{Rep}A_2. \tag{8}$$

Now, A' is a convex fuzzy set which has the same representative value as A^*. The proof is ignored here, interested readers may refer to [1]. Similarly, the consequent fuzzy set B' can be obtained by

$$B' = (1 - \lambda_{Rep})B_1 + \lambda_{Rep}B_2. \tag{9}$$

In so doing, the newly derived rule $A' \Rightarrow B'$ involves the use of only normal and convex fuzzy sets.

As $A' \Rightarrow B'$ is derived from $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$, it is feasible to perform fuzzy reasoning with this new rule without further reference to its originals. The interpolative reasoning problem is therefore changed from (4) to the new *modus ponens* interpretation:

$$\begin{array}{l} \text{observation: } X \text{ is } A^* \\ \text{rule: if } X \text{ is } A', \text{ then } Y \text{ is } B' \\ \hline \text{conclusion: } Y \text{ is } B^*? \end{array} \tag{10}$$

This interpretation retains the same results as (4) in dealing with the extreme cases: If $A^* = A_1$, then from (6) $\lambda_{Rep} = 0$, and according to (8) and (9), $A' = A_1$ and $B' = B_1$, so the conclusion $B^* = B_1$. Similarly, if $A^* = A_2$, then $B^* = B_2$.

Other than the extreme cases, *similarity* measures are used to support the application of this new *modus ponens*. In particular, (10) can be interpreted as

The more similar X to A', the more similar Y to B'. (11)

Suppose that a certain degree of similarity between A' and A^* is established, it is intuitive to require that the consequent parts B' and B^* attain the same similarity degree. The question is how to obtain an operator which can represent the similarity degree between A' and A^*, and how to transform B' to B^* with the desired degree of similarity. To this end, the following two (component) transformations are proposed. The difference between the present method and the work of [1], [13] rests in the implementation of these two transformations.

B. Scale Transformation

Consider applying scale transformation to an arbitrary polygonal fuzzy membership function $A = (a_0, \ldots, a_{n-1})$ (as shown in Fig. 3) to generate $A' = (a'_0, \ldots, a'_{n-1})$, such that they have the same RV, and $a'_{n-1-i} - a'_i = s_i(a_{n-1-i} - a_i)$, where s_i are scale rates and $i = \{0, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$. In order to achieve this, $\lfloor \frac{n}{2} \rfloor$ equations $a'_{n-1-i} - a'_i = s_i(a_{n-1-i} - a_i)$, $i = \{0, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$, are imposed to obtain the supports with desired lengths, and $(\lceil \frac{n}{2} \rceil - 1)$ equations $\frac{a'_{i+1} - a'_i}{a'_{n-1-i} - a'_{n-2-i}} = \frac{a_{i+1} - a_i}{a_{n-1-i} - a_{n-2-i}}$, $i = \{0, \ldots, \lceil \frac{n}{2} \rceil - 2\}$ are
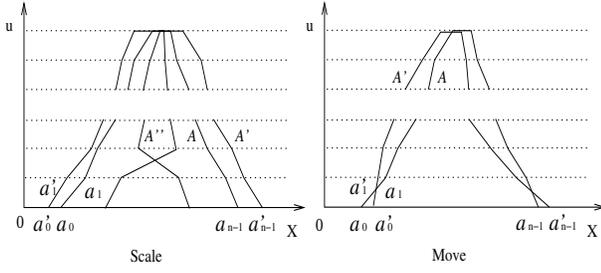
Fig. 3. Scale and move transformations

imposed to equalise the ratios between the left ($\lceil \frac{n}{2} \rceil - 1$) slopes' lengths and the right ($\lceil \frac{n}{2} \rceil - 1$) slopes' lengths of A' to their counterparts of the original fuzzy set A. The equation $\sum_{i=0}^{n-1} w_i a_i' = \sum_{i=0}^{n-1} w_i a_i$ which ensures the same representative values before and after the transformation is added to make up of $\lfloor \frac{n}{2} \rfloor + (\lceil \frac{n}{2} \rceil - 1) + 1 = n$ equations. All these n equations are collectively written as:

$$
\begin{cases}
a_{n-1-i}' - a_i' = s_i(a_{n-1-i} - a_i) = S_i \\
(i = \{0, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}) \\
\frac{a_{i+1}' - a_i'}{a_{n-1-i}' - a_{n-2-i}'} = \frac{a_{i+1} - a_i}{a_{n-1-i} - a_{n-2-i}} = R_i \\
(i = \{0, \ldots, \lceil \frac{n}{2} \rceil - 2\}) \\
\sum_{i=0}^{n-1} w_i a_i' = \sum_{i=0}^{n-1} w_i a_i
\end{cases}
\tag{12}
$$

where S_i is the i-th support length of the resultant fuzzy set and R_i is the ratio between the left i-th slope length and the right i-th slope length. Solving these n equations simultaneously results in an unique and convex fuzzy set A' given that the resultant set has the descending order of the support lengths from the bottom to the top.

So far the proposed scale transformation remains the same as the original work of [1], the difference is the way of calculating scale rates. Recall that the scale ratios $\mathbb{S}$ are introduced in the original scale transformations, to ensure the support lengths decreasing from the bottom support to top support. Instead of doing that, *left scale factor* $\mathbb{SL}_i$ and *right scale factor* $\mathbb{SR}_i$ are introduced for the i-th support, $i = \{0, \ldots, \lceil \frac{n}{2} \rceil - 2\}$.

$$
\mathbb{SL}_i = \frac{a_{i+1}' - a_i'}{a_{i+1} - a_i},
\tag{13}
$$

$$
\mathbb{SR}_i = \frac{a_{n-1-i}' - a_{n-2-i}'}{a_{n-1-i} - a_{n-2-i}}.
\tag{14}
$$

Obviously, $\mathbb{SL}_i \geq 0$ and $\mathbb{SR}_i \geq 0$ if both A and A' are convex. Having introduced these, the scale rate of the i-th support is computed so that

$$
\begin{aligned}
s_i &= \frac{S_i'}{S_i} = \frac{a_{n-1-i}' - a_i'}{a_{n-1-i} - a_i} \\
&= \frac{U + a_{n-2-i}' - a_{i+1}' + V}{a_{n-1-i} - a_i} \\
&= \frac{U + s_{i+1}(a_{n-2-i} - a_{i+1}) + V}{a_{n-1-i} - a_i},
\end{aligned}
\tag{15}
$$

where S_i' and S_i are the lengths of the i-th support of A' and A respectively, $U = \mathbb{SL}_i(a_{i+1} - a_i)$ and $V = \mathbb{SR}_i(a_{n-1-i} - a_{n-2-i})$. As $S_i' = S_{i+1}' + \mathbb{SL}_i(a_{i+1} - a_i) + \mathbb{SR}_i(a_{n-1-i} -$

$a_{n-2-i})$, if $\mathbb{SL}_i \geq 0$ and $\mathbb{SR}_i \geq 0$, then $\mathbb{SL}_i(a_{i+1} - a_i) \geq 0$ and $\mathbb{SR}_i(a_{n-1-i} - a_{n-2-i}) \geq 0$, hence $S_i' \geq S_{i+1}'$ must hold. So the scale transformation guarantees to generate an NCF fuzzy set.

Conversely, if two convex sets $A = (a_0, \ldots, a_{n-1})$ and $A' = (a_0', \ldots, a_{n-1}')$ which have the same RV are given, the left and right scale factors of the i-th support, $\mathbb{SL}_i$, $\mathbb{SR}_i$ ($i = \{0, \ldots, \lceil \frac{n}{2} \rceil - 2\}$) can be calculated by (13) and (14) respectively. Given that A and A' are both convex, $\mathbb{SL}_i \geq 0$ and $\mathbb{SR}_i \geq 0$ must hold.

Special treatments are needed if: 1) A has a vertical left slope on the i-th support level, the term of $(a_{i+1} - a_i)$ in (13) is replaced by the vertical distance of the i-th and $(i+1)$-th points to avoid the division by zero; and 2) A has a vertical right slope on the i-th support level, the term of $(a_{n-1-i} - a_{n-2-i})$ in (14) is replaced by the vertical distance of the i-th and $(i+1)$-th points.

The above scale factors are calculated from top to bottom (so are the scale rates). If on the contrary, the calculation order is from bottom to top, then it would be possible that the scaled fuzzy set becomes non-convex, as A'' of Fig. 3.

C. Move Transformation

The proposed move transformation is no longer like the original (see [1]). Instead, it appears rather like the original scale transformation.

After performing the scale transformation, the lengths of supports of a fuzzy set become equal to those of the desired fuzzy set (A'). Now the move transformation is used to move the supports to appropriate positions. Consider applying the move transformation to an arbitrary polygonal fuzzy membership function $A = (a_0, \ldots, a_{n-1})$ (as shown in Fig. 3) to generate $A' = (a_0', \ldots, a_{n-1}')$ such that they have the same RV and the same lengths of supports. In order to achieve this, $\lfloor \frac{n}{2} \rfloor$ equations $a_{n-1-i}' - a_i' = a_{n-1-i} - a_i$, $i = \{0, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$, are imposed to ensure the same lengths of supports, and ($\lceil \frac{n}{2} \rceil - 1$) equations $\frac{a_{i+1}' - a_i'}{a_{n-1-i}' - a_{n-2-i}'} / \frac{a_{i+1} - a_i}{a_{n-1-i} - a_{n-2-i}} = \mathbb{RC}_i$ (where $i = \{0, \ldots, \lceil \frac{n}{2} \rceil - 2\}$ and $\mathbb{RC}_i$ is the *move factor* for the i-th support) are imposed to set the ratios between the i-th left slope length and the i-th right slope length of A', to their counterparts of the original fuzzy set A. The equation $\sum_{i=0}^{n-1} w_i a_i' = \sum_{i=0}^{n-1} w_i a_i$ which ensures the same representative values before and after the transformation is added to make up of $\lfloor \frac{n}{2} \rfloor + (\lceil \frac{n}{2} \rceil - 1) + 1 = n$ equations. All these n equations are collectively written as:

$$
\begin{cases}
a_{n-1-i}' - a_i' = a_{n-1-i} - a_i = S_i \\
(i = \{0, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}) \\
\frac{a_{i+1}' - a_i'}{a_{n-1-i}' - a_{n-2-i}'} / \frac{a_{i+1} - a_i}{a_{n-1-i} - a_{n-2-i}} = \mathbb{RC}_i \\
(i = \{0, \ldots, \lceil \frac{n}{2} \rceil - 2\}) \\
\sum_{i=0}^{n-1} w_i a_i' = \sum_{i=0}^{n-1} w_i a_i
\end{cases}
\tag{16}
$$

where S_i is the i-th support length of the fuzzy set (either before or after moving). If $\mathbb{RC}_i \geq 0$, solving these n equations simultaneously results in an unique and convex fuzzy set.

Conversely, if two convex sets $A = (a_0, \ldots, a_{n-1})$ and $A' = (a_0', \ldots, a_{n-1}')$ are given, which have the same RV and the same lengths of supports, the move factor of the i-th support, $\mathbb{RC}_i$ ($i = \{0, \ldots, \lceil \frac{n}{2} \rceil - 2\}$) can be calculated by (16). Given that A and A' are both convex, $\mathbb{RC}_i \geq 0$ must hold.

Unlike the scale transformations, the move transformation does not have to follow a fixed order for calculation. That is, it does not matter whether the calculation is carried out from the top to the bottom or otherwise. However, there are special cases which need extra consideration in calculating the move factor: 1) If A' has a vertical right slope on the i-th support level, the move factor is set to -1 in the implementation. When any fuzzy sets are moved with such a move factor, they become fuzzy sets of a vertical right slope on the i-th support level. 2) If the original fuzzy set A has a vertical left slope on the i-th support level, the term $(a_{i+1} - a_i)$ will be replaced by the vertical distance between the i-th and $(i+1)$-th points. 3) If A has a vertical right slope on the i-th support level, the term $(a_{n-i-1} - a_{n-i-2})$ will be replaced by the vertical distance between the i-th and $(i+1)$-th points. These are needed to avoid division by zero.

D. Summary

As indicated earlier, it is intuitive to maintain the similarity degree between the consequent parts $B' = (b'_0, \ldots, b'_{n-1})$ and $B^* = (b^*_0, \ldots, b^*_{n-1})$ to be the same as that between the antecedent parts $A' = (a'_0, \ldots, a'_{n-1})$ and $A^* = (a^*_0, \ldots, a^*_{n-1})$. In this work, the proposed scale and move transformations indeed allow such similarity degrees to be held by the use of the *scale* and *move factors*. In summary, the desired conclusion B^* can be obtained as follows:

1) Calculate scale rates s_i $(i = \{0, 1 \ldots, \lfloor \frac{n}{2} \rfloor - 1\})$ of the i-th support from A' to A^* according to $s_i = \frac{a^*_{n-1-i} - a^*_i}{a'_{n-1-i} - a'_i}$.
2) Apply scale transformation to A' using scale rates s_i $(i = \{0, 1 \ldots, \lfloor \frac{n}{2} \rfloor - 1\})$ as computed above to obtain A'', by simultaneously solving n linear equations as given in (12).
3) Calculate left and right scale factors $\mathbb{SL}_i$, $\mathbb{SR}_i$, $i = \{0 \ldots, \lceil \frac{n}{2} \rceil - 2\}$, of the i-th support from A' to A^* according to (13) and (14).
4) Calculate scale rates s'_i $(i = \{0, 1 \ldots, \lceil \frac{n}{2} \rceil - 2\})$ of the i-th support from B' to B^* according to (15). Note that if B' has two points of membership value 1, $s'_{\lfloor \frac{n}{2} \rfloor - 1} = s_{\lfloor \frac{n}{2} \rfloor - 1}$.
5) Apply scale transformation to B' using s'_i $(i = \{0, 1 \ldots, \lfloor \frac{n}{2} \rfloor - 1\})$ as calculated in step 4 to obtain $B'' = (b''_0, \ldots, b''_{n-1})$, by simultaneously solving the n linear equations as shown in (12).
6) Calculate move factor $\mathbb{RC}_i$, $i = \{0, \ldots, \lceil \frac{n}{2} \rceil - 2\}$, on the i-th support level from A'' to A^* according to (16).
7) Apply move transformation to B'' using the move factor as calculated in step 6 to obtain B^*, by simultaneously solving the n linear equations as shown in (16).

Clearly, B' and B^* will have the same similarity degree as that between the antecedent parts A' and A^*.

The interpolation involving multiple antecedent variables or multiple rules has been similarly extended. Interested readers may refer to [13] for detailed discussion.

IV. PRESERVATION OF PIECEWISE LINEARITY

Preservation of piecewise linearity is an essential property which reflects how good the interpolative reasoning method handles the points between two consecutive α-cut levels. If piecewise linearity is preserved, it is safe to merely consider the characteristic points rather than the infinite pairs of points (generated from infinite α-cut levels). The preservation of

piecewise linearity has been investigated in the work of [14], [15]. In both cases, they slightly deviate from the real linear fuzzy rule interpolation with given error bounds. This section shows that the proposed method preserves this property in interpolations involving arbitrary polygonal fuzzy membership functions.

Fig. 4 illustrates the scale transformation in a trapezoidal case with six characteristic points for each fuzzy set. Suppose that $a^s_{0\alpha}$, $a^s_{3\alpha}$, $a'_{0\alpha}$, $a'_{3\alpha}$, $b'_{0\alpha}$ and $b'_{3\alpha}$ are artificial characteristic points. If B^s is transformed from B' using the same similarity calculated from A' to A^s, the question is whether $b^s_{0\alpha}$ and $b^s_{3\alpha}$ remain artificial. According to the proposed method,

$$\frac{a^s_1 - a^s_{0\alpha}}{a'_1 - a'_{0\alpha}} = \frac{b^s_1 - b^s_{0\alpha}}{b'_1 - b'_{0\alpha}}, \tag{17}$$

$$\frac{a^s_{0\alpha} - a^s_0}{a'_{0\alpha} - a'_0} = \frac{b^s_{0\alpha} - b^s_0}{b'_{0\alpha} - b'_0}. \tag{18}$$

Also, as $a'_{0\alpha}$ and $a^s_{0\alpha}$ are two artificial characteristic points, then

$$\frac{a^s_1 - a^s_{0\alpha}}{a'_1 - a'_{0\alpha}} = \frac{a^s_{0\alpha} - a^s_0}{a'_{0\alpha} - a'_0}. \tag{19}$$

From (17), (18) and (19),

$$\frac{b^s_1 - b^s_{0\alpha}}{b'_1 - b'_{0\alpha}} = \frac{b^s_{0\alpha} - b^s_0}{b'_{0\alpha} - b'_0}. \tag{20}$$
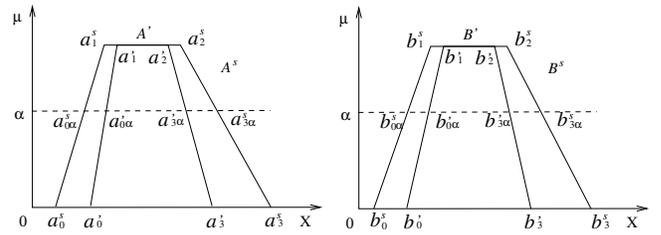
Fig. 4. Preservation of piecewise linearity in scale transformation

From (20) and the fact that $b'_{0\alpha}$ is an artificial characteristic point, it can be concluded that $b^s_{0\alpha}$ must be artificial. That is, B^s is piecewise linear in the left slope. Similarly, B^s is piecewise linear in the right slope. Thus, the proposed method preserves the piecewise linearity in performing scale transformations. Although the proof is based on the trapezoidal cases, it can be straightforwardly extended to any scale transformation involving arbitrary polygonal fuzzy membership functions.

Now, consider the move transformation which is shown in Fig. 5. Given A^s and A^* which have the same RV and the same lengths of top, middle and bottom supports respectively, the task is to move B^s in order to obtain B^* such that they have the same similarity as that between A^s and A^*. According to the enhanced move transformation,

$$\frac{a^*_1 - a^*_{0\alpha}}{a^*_{3\alpha} - a^*_2} \frac{a^s_{3\alpha} - a^s_2}{a^s_1 - a^s_{0\alpha}} = \frac{b^*_1 - b^*_{0\alpha}}{b^*_{3\alpha} - b^*_2} \frac{b^s_{3\alpha} - b^s_2}{b^s_1 - b^s_{0\alpha}}, \tag{21}$$

$$\frac{a^*_{0\alpha} - a^*_0}{a^*_3 - a^*_{3\alpha}} \frac{a^s_3 - a^s_{3\alpha}}{a^s_{0\alpha} - a^s_0} = \frac{b^*_{0\alpha} - b^*_0}{b^*_3 - b^*_{3\alpha}} \frac{b^s_3 - b^s_{3\alpha}}{b^s_{0\alpha} - b^s_0}. \tag{22}$$

Assume that $a^s_{0\alpha}$, $a^*_{0\alpha}$, $a^s_{3\alpha}$ and $a^*_{3\alpha}$ are arbitrary characteristic points, then

$$\frac{a^*_1 - a^*_{0\alpha}}{a^*_{3\alpha} - a^*_2} \frac{a^s_{3\alpha} - a^s_2}{a^s_1 - a^s_{0\alpha}} = \frac{a^*_{0\alpha} - a^*_0}{a^*_3 - a^*_{3\alpha}} \frac{a^s_3 - a^s_{3\alpha}}{a^s_{0\alpha} - a^s_0}. \tag{23}$$
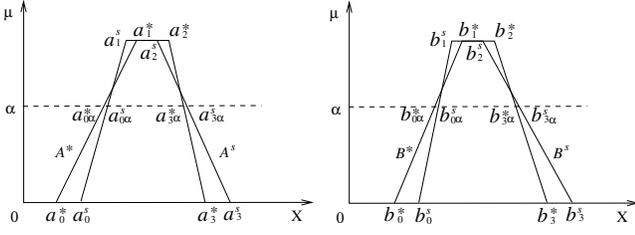
Fig. 5. Preservation of piecewise linearity in move transformation

From (21), (22) and (23),

$$\frac{b_1^* - b_{0\alpha}^*}{b_{3\alpha}^* - b_2^*}\frac{b_{3\alpha}^s - b_2^s}{b_1^s - b_{0\alpha}^s} = \frac{b_{0\alpha}^* - b_0^*}{b_3^* - b_{3\alpha}^*}\frac{b_3^s - b_{3\alpha}^s}{b_{0\alpha}^s - b_0^s}. \tag{24}$$

Given that $b_{0\alpha}^s$ and $b_{3\alpha}^s$ are artificial characteristic points, it follows that

$$\frac{b_{3\alpha}^s - b_2^s}{b_1^s - b_{0\alpha}^s} = \frac{b_3^s - b_{3\alpha}^s}{b_{0\alpha}^s - b_0^s}. \tag{25}$$

From (24) and (25),

$$\frac{b_1^* - b_{0\alpha}^*}{b_{3\alpha}^* - b_2^*} = \frac{b_{0\alpha}^* - b_0^*}{b_3^* - b_{3\alpha}^*}. \tag{26}$$

As $b_2^s - b_1^s = b_2^* - b_1^*$, $b_{3\alpha}^s - b_{0\alpha}^s = b_{3\alpha}^* - b_{0\alpha}^*$ and $b_3^s - b_0^s = b_3^* - b_0^*$, it can be concluded that $b_{0\alpha}^*$ and $b_{3\alpha}^*$ are artificial. The proof also applies to move transformations involving arbitrary polygonal fuzzy membership functions. Thus, the piecewise linearity is preserved in performing move transformation.

It has been established in [1] that the step of constructing the intermediate rule preserves piece-wise linearity. Now that it has proven that the scale and move transformations preserve piece-wise linearity, property 1 as shown below is achieved by the proposed interpolation method:

Property 1: The interpolation of non-characteristic points which lie between two characteristic points generates non-characteristic points which still lie between the two interpolated characteristic points.

This property points out that only characteristic points affect the interpolated results using the proposed method. Non-characteristic points can be safely ignored as they are still non-characteristic in the reasoning results.

If the representative value of a fuzzy set keeps the same when more artificial characteristic points are considered in the proposed interpolation, then the following property holds:

Property 2: The interpolation over two fuzzy sets that are otherwise the same except that one has additional artificial characteristic points leads to the same result if the representative values used by them are the same.

Note that the work of [11], [12] represents each fuzzy set with n characteristic points as a point in an n-dimensional Cartesian space. In this case, fuzzy interpolation becomes a high dimensional interpolation problem. Since the newly proposed method is capable of handling fuzzy interpolation involving infinite points (finite characteristic points plus infinite non-characteristic points), it may provide a solution to the interpolation problem within a very high dimensional (or even infinite) Cartesian space.

V. Illustrative Examples

In this section, the use of the average RV, weighted average RV and centre of core RV respectively to conduct fuzzy interpolation is demonstrated and the results between the work of [1] and the newly proposed method are compared. For simplicity, both examples discussed below concern the interpolation between two adjacent rules $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$. In order to verify the piece-wise linearity property, additional "characteristic" points are deliberately added in the examples. Table I shows the fuzzy values of the rule attributes and observations. Table II and Table III show the interpolated results using different RV definitions for the work of [1] (denoted as original HS method) and the newly proposed method (denoted as new HS method), respectively. These results are also illustrated in Fig. 6 and Fig. 7. As can be seen, the original interpolation method satisfies property 1 only in triangular cases while the newly proposed method satisfies that in all cases. In particular, the latter further holds property 2 when the center core representative value is used. As a comparison, the results of the KH method is also given in Fig. 6. It satisfies neither property 1 nor property 2.
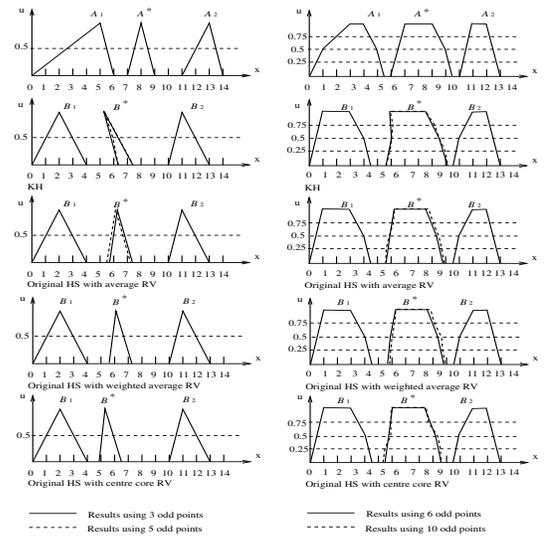

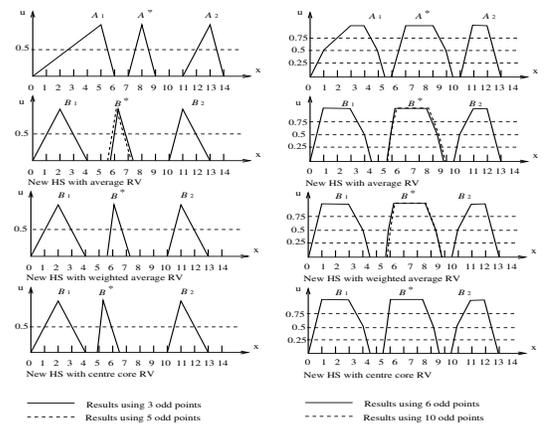Fig. 6. Examples of piecewise linearity for KH and original HS method

Fig. 7. Examples of piecewise linearity for new HS method

TABLE I

ATTRIBUTE AND OBSERVATION VALUES

	Triangular	Triangular (5 points)	Hexagonal	Hexagonal (8 points)
A_1	$(0, 5, 6)$	$(0, 2.5, 5, 5.5, 6)$	$(0, 1, 3,$ $4, 5, 5.5)$	$(0, 0.5, 1, 2, 3,$ $4, 4.5, 5, 5.25, 5.5)$
A_2	$(11, 13, 14)$	$(11, 12, 13, 13.5, 14)$	$(11, 11.5, 12,$ $13, 13.5, 14)$	$(11, 11.25, 11.5, 11.75, 12,$ $13, 13.25, 13.5, 13.75, 14)$
A^*	$(7, 8, 9)$	$(7, 7.5, 8, 8.5, 9)$	$(6, 6.5, 7,$ $9, 10, 10.5)$	$(6, 6.25, 6.5, 6.75, 7,$ $9, 9.5, 10, 10.25, 10.5)$
B_1	$(0, 2, 4)$	$(0, 1, 2, 3, 4)$	$(0, 0.5, 1,$ $3, 4, 4.5)$	$(0, 0.25, 0.5, 0.75, 1,$ $3, 3.5, 4, 4.25, 4.5)$
B_2	$(10, 11, 13)$	$(10, 10.5, 11, 12, 13)$	$(10.5, 11, 12,$ $13, 13.5, 14)$	$(10.5, 10.75, 11, 11.5, 12,$ $13, 13.25, 13.5, 13.75, 14)$

TABLE II

RESULTS OF ORIGINAL HS METHOD BY USING DIFFERENT RVS

	Triangular	Triangular (5 points)	Hexagonal	Hexagonal (8 points)
Average	$(5.84, 6.26,$ $7.38)$	$(5.76, 5.97, 6.18,$ $6.74, 7.3)$	$(5.64, 5.98, 6.29,$ $8.63, 9.46, 9.93)$	$(5.64, 5.82, 5.98, 6.14, 6.28,$ $8.64, 9.05, 9.47, 9.71, 9.94)$
Weighted Average	$(5.63, 6.06,$ $7.16)$	$(5.63, 5.85, 6.06,$ $6.61, 7.16)$	$(5.61, 5.95, 6.26,$ $8.59, 9.42, 9.89)$	$(5.62, 5.80, 5.96, 6.11, 6.26,$ $8.62, 9.02, 9.44, 9.68, 9.91)$
Center of Core	$(4.96, 5.38,$ $6.44)$	$(4.96, 5.17, 5.38,$ $5.91, 6.44)$	$(5.47, 5.79, 6.08$ $8.42, 9.23, 9.70)$	$(5.46, 5.64, 5.81, 5.95, 6.07,$ $8.43, 8.83, 9.25, 9.47, 9.70)$

TABLE III

RESULTS OF NEW HS METHOD BY USING DIFFERENT RVS

	Triangular	Triangular (5 points)	Hexagonal	Hexagonal (8 points)
Average	$(5.54, 5.97,$ $7.97)$	$(5.49, 5.70, 5.92,$ $6.92, 7.92)$	$(5.28, 5.62, 5.94,$ $8.86, 9.86, 10.36)$	$(5.28, 5.45, 5.62, 5.79, 5.95,$ $8.87, 9.37, 9.87, 10.12, 10.37)$
Weighted Average	$(5.41, 5.83,$ $7.83)$	$(5.41, 5.62, 5.83,$ $6.83, 7.83)$	$(5.25, 5.59, 5.91,$ $8.85, 9.85, 10.35)$	$(5.26, 5.43, 5.60, 5.72, 5.92,$ $8.85, , 9.35, 9.85, 10.10, 10.35)$
Center of Core	$(4.96, 5.38,$ $7.38)$	$(4.96, 5.17, 5.38,$ $6.38, 7.38)$	$(5.12, 5.45, 5.75$ $8.75, 9.75, 10.25)$	$(5.12, 5.28, 5.45, 5.60, 5.75,$ $8.75, 9.25, 9.75, 10.00, 10.25)$

VI. CONCLUSIONS

This paper, based on the work of [1], has proposed an interpolative method which preserves the piece-wise linearity for interpolations involving arbitrary polygonal fuzzy sets. In so doing, the non-characteristic points can be safely ignored in performing interpolation. In addition, the newly proposed method inherits the advantages of its original: 1) it can easily handle interpolation (or even extrapolation, see [13]) of multiple antecedent variables or multiple rules with simple computation; 2) it guarantees the uniqueness as well as normality and convexity of the resulting interpolated fuzzy sets; and 3) it provides a degree of freedom to choose different RVs for application requirements. Although theoretically the proposed approach offers promising potential, it requires further evaluation of how it may perform when presented with a complex real-world problem. As with other interpolation techniques, such a problem may involve rule base simplification or reasoning with a sparse rule base. Work is on-going to identify problems of this nature, in order to better assess the practical limitation of the method.

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