Fuzzy Interpolative Reasoning via Scale and Move Transformations

Zhiheng Huang and Qiang Shen

***Abstract*—Interpolative reasoning does not only help reduce the complexity of fuzzy models but also makes inference in sparse rule-based systems possible. This paper presents an interpolative reasoning method by means of scale and move transformations. It can be used to interpolate fuzzy rules involving complex polygon, Gaussian or other bell-shaped fuzzy membership functions. The method works by first constructing a new inference rule via manipulating two given adjacent rules, and then by using scale and move transformations to convert the intermediate inference results into the final derived conclusions. This method has three advantages thanks to the proposed transformations: 1) it can handle interpolation of multiple antecedent variables with simple computation; 2) it guarantees the uniqueness as well as normality and convexity of the resulting interpolated fuzzy sets; and 3) it suggests a variety of definitions for representative values, providing a degree of freedom to meet different requirements. Comparative experimental studies are provided to demonstrate the potential of this method.**

***Index Terms*—Fuzzy model simplification, fuzzy rule interpolation, scale and move transformations, sparse rule base, transformation-based interpolation.**

I. Introduction

FUZZY rule interpolation helps reduce the complexity of fuzzy models and supports inference in systems that employ sparse rule sets [7], [10]. With interpolation, fuzzy rules which may be approximated from their neighboring rules can be omitted from the rule base. This leads to the complexity reduction of fuzzy models. When given observations have no overlap with the antecedent values of the rules, classical fuzzy inference methods have no rule to fire, but interpolative reasoning methods can still obtain certain conclusions. Despite these significant advantages, earlier work in fuzzy interpolative reasoning does not guarantee the convexity of the derived fuzzy sets [12], [17], which is often a crucial requirement of fuzzy reasoning to attain more easily interpretable practical results.

In order to eliminate the nonconvexity drawback, there has been considerable work reported in the literature. For instance, Vas *et al.* have proposed an algorithm [14] that reduces the problem of nonconvex conclusions. Qiao *et al.* [11] have published an improved method which uses similarity transfer reasoning to guarantee the attainment of convex results. Hsiao *et al.* [4] have introduced a new interpolative method which exploits the slopes of the fuzzy sets. General fuzzy interpolation and extrapolation techniques [1], and a modified α-cut based method [2], [13], have also been proposed. In addition, Bouchon *et al.* have created an interpolative method by exploiting the concept of graduality [3], and Yam and Kóczy [15], [16] have proposed a fuzzy interpolative technique based on Cartesian representation.

Nevertheless, some of the existing methods include complex computation. It becomes more difficult when they are extended to multiple variables interpolation. Others may only apply to simple fuzzy membership functions limited to triangular or trapezoidal. Almost all existing techniques lack the flexibility to generate results that meet different application requirements, whilst the work of [15] and [16] generates multiple reasoning results without showing how to make a choice amongst them. This paper, based on the initial work carried out by the authors [5], [6], proposes a novel interpolative reasoning method which avoids the problems mentioned above. First, intermediate fuzzy rules are constructed by their adjacent rules. These, together with the observations, are then converted into the final fuzzy consequences by the scale and move transformations, ensuring unique, normal and convex results in an elegant manner.

The rest of the paper is organized as follows. Section II reviews the relevant background of fuzzy interpolative techniques. Section III describes the new interpolative reasoning method. Section IV gives examples to illustrate the use and power of this method by comparing it against typical existing approaches. Finally, Section V concludes the paper and points out important further work.

II. Background of Fuzzy Rule Interpolative Techniques

Fuzzy rule interpolation [7]–[9], proposed first by Kóczy and Hirota, is an inference technique for fuzzy rule bases where the antecedents do not cover the whole input universe. Such techniques are essential for sparse rule-based fuzzy systems. The initial rule interpolation method, which is hereafter referred to as the KH algorithm for presentational simplicity, requires the following conditions to be satisfied: The involved fuzzy sets have to be of continuous, normal and convex membership functions, with bounded support. This is not so restrictive as it might sound as such fuzzy sets are those typically used in both theoretical and practical fuzzy systems.

An important notion in [7] is the "less than" relation between two fuzzy sets. Fuzzy set A_1 is said to be less than A_2, denoted by $A_1 \prec A_2$, if $\forall \alpha \in [0,1]$, the following conditions hold:

$$\inf\{A_{1\alpha}\} < \inf\{A_{2\alpha}\},\ \sup\{A_{1\alpha}\} < \sup\{A_{2\alpha}\} \qquad (1)$$

where $A_{1\alpha}$ and $A_{2\alpha}$ are, respectively, the α-cut of A_1 and that of A_2, $\inf\{A_{i\alpha}\}$ is the infimum of $A_{i\alpha}$, and $\sup\{A_{i\alpha}\}$ is the supremum of $A_{i\alpha}$, $i = 1, 2$.

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Z. Huang is with the School of Informatics, University of Edinburgh, Edinburgh EH8 9LE, U.K. (e-mail: Z.huang-2@sms.ed.ac.uk).

Q. Shen is with the Department of Computer Science, University of Wales, Aberystwyth SY23 3DB, U.K. (e-mail: qqs@aber.ac.uk).

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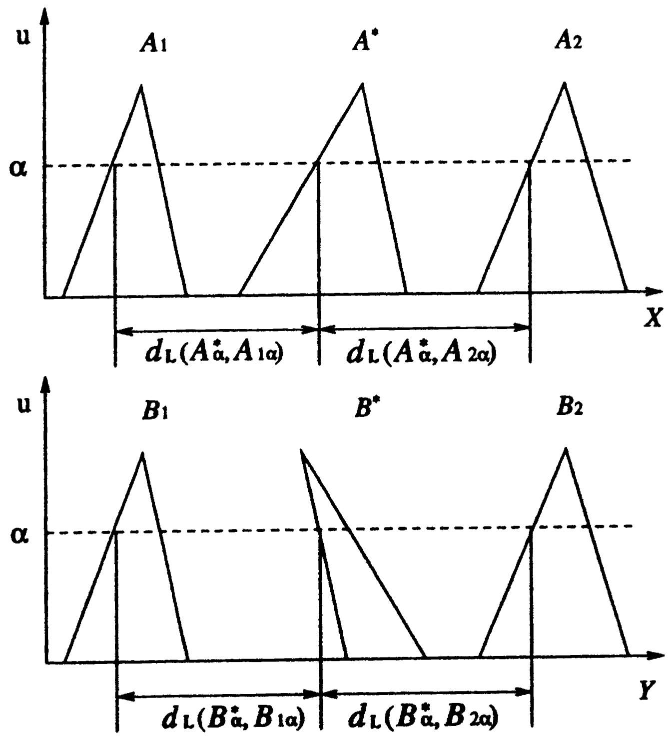

Fig. 1. Fuzzy interpolative reasoning with a nonconvex conclusion on a sparse fuzzy rule base.

For simplicity, suppose that two fuzzy rules are given

If X is A_1 then Y is B_1
If X is A_2 then Y is B_2

which are briefly denoted as $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$, respectively. Also, suppose that these two rules are adjacent, i.e., there is no any such a rule existing that the antecedent value A of that rule is between the regions of A_1 and A_2. To entail the interpolation between the consequent values of these two rules, i.e., to determine a new conclusion B^* when an observation A^* located between fuzzy sets A_1 and A_2 is given, it is commonly assumed, for convenience, that rules in a given rule base are arranged with respect to a partial ordering among the normal and convex fuzzy sets (NCF sets) of the antecedents variables. For the previous two rules, this means that

$$A_1 \prec A^* \prec A_2. \tag{2}$$

The simplest interpolation which is linear can thus be written as

$$\frac{d(A^*, A_1)}{d(A^*, A_2)} = \frac{d(B^*, B_1)}{d(B^*, B_2)} \tag{3}$$

where $d(.,.)$ is typically the Euclidean distance between two fuzzy sets (though other distance metrics may be used as alternatives for this). This is illustrated in Fig. 1, where the lower and upper distances between α-cuts $A_{1\alpha}$ and $A_{2\alpha}$ are defined as follows:

$$d_L(A_{1\alpha}, A_{2\alpha}) = d\left(\inf\{A_{1\alpha}\}, \inf\{A_{2\alpha}\}\right) \tag{4}$$
$$d_U(A_{1\alpha}, A_{2\alpha}) = d\left(\sup\{A_{1\alpha}\}, \sup\{A_{2\alpha}\}\right). \tag{5}$$

From (4) and (5), (3) can be rewritten as

$$\min\{B^*_\alpha\} = \frac{\frac{\inf\{B_{1\alpha}\}}{d_L(A^*_\alpha, A_{1\alpha})} + \frac{\inf\{B_{2\alpha}\}}{d_L(A^*_\alpha, A_{2\alpha})}}{\frac{1}{d_L(A^*_\alpha, A_{1\alpha})} + \frac{1}{d_L(A^*_\alpha, A_{2\alpha})}} \tag{6}$$
$$\max\{B^*_\alpha\} = \frac{\frac{\sup\{B_{1\alpha}\}}{d_U(A^*_\alpha, A_{1\alpha})} + \frac{\sup\{B_{2\alpha}\}}{d_U(A^*_\alpha, A_{2\alpha})}}{\frac{1}{d_U(A^*_\alpha, A_{1\alpha})} + \frac{1}{d_U(A^*_\alpha, A_{2\alpha})}}. \tag{7}$$

Alternatively, let

$$\lambda_L = \frac{d_L(A^*_\alpha, A_{1\alpha})}{d_L(A_{2\alpha}, A_{1\alpha})} \quad \lambda_U = \frac{d_U(A^*_\alpha, A_{1\alpha})}{d_U(A_{2\alpha}, A_{1\alpha})}. \tag{8}$$

The same solution can then be obtained but represented differently as follows:

$$\min\{B^*_\alpha\} = (1 - \lambda_L)\inf\{B_{1\alpha}\} + \lambda_L \inf\{B_{2\alpha}\} \tag{9}$$
$$\max\{B^*_\alpha\} = (1 - \lambda_U)\sup\{B_{1\alpha}\} + \lambda_U \sup\{B_{2\alpha}\}. \tag{10}$$

From this, $B^*_\alpha = (\min\{B^*_\alpha\}, \max\{B^*_\alpha\})$ results. From B^*_α, in turn, the conclusion fuzzy set B^* can be constructed by the representation principle of fuzzy sets:

$$B^* = \bigcup_{\alpha \in [0,1]} \alpha B^*_\alpha. \tag{11}$$

However, this linear interpolation cannot guarantee the convexity of the derived fuzzy sets (although they may be normal, as shown in Fig. 1), even when the fuzzy sets concerned in both the given rules and the observations are all convex and normal. Such work may even return a conclusion that is not possible to be represented as a fuzzy membership function (see Example 1 in Section V). Thus, much work remains to improve such an interpolation method to ensure not only normality but also convexity of inferred consequences.

III. The Proposed Method

A. Single Antecedent Variable With Triangular Fuzzy Sets

Triangular fuzzy membership functions are firstly considered to demonstrate the basic ideas of the present work, due to its simplicity and popularity. This is to be followed by more complex functions such as trapezoidal and Gaussian in the next subsections. Also for presentational simplicity, only rules involving one antecedent variable are dealt with here, with a generalized case to be given later.

To facilitate this discussion, the *representative value* of a triangular membership function is defined as the average of the x coordinates of its three key points: the left and right extreme points (whose membership values are 0) and the normal point (whose membership value is 1). Without losing generality, given a fuzzy set A, denoted as (a_0, a_1, a_2), as shown in Fig. 2, its representative value is

$$\text{Rep}(A) = \frac{a_0 + a_1 + a_2}{3}. \tag{12}$$

This representative value happens to be the x coordinate of the centre of gravity of such a triangular fuzzy set [5].

Suppose that two adjacent fuzzy rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ and the observation A^*, which is located between fuzzy sets

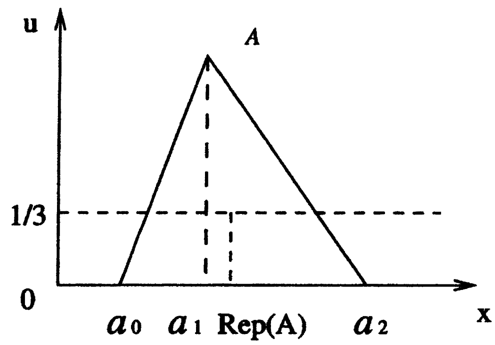

Fig. 2. Representative value of a triangular fuzzy set.

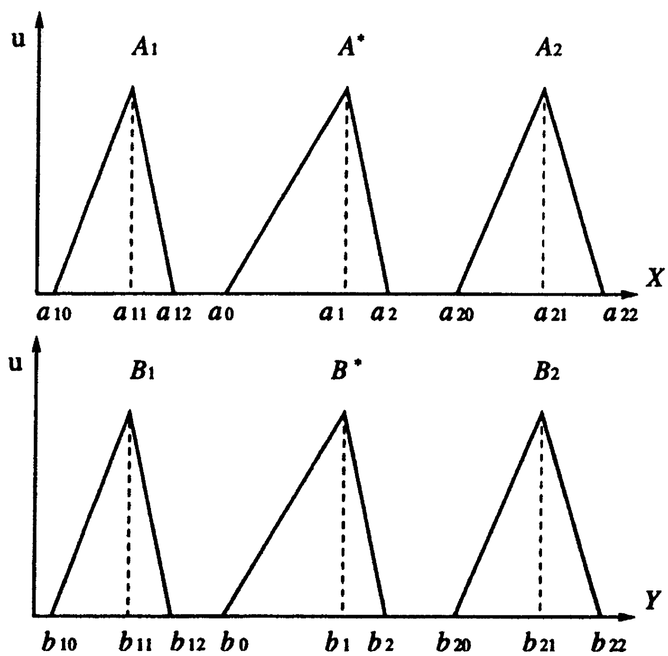

Fig. 3. Interpolation with triangular membership functions.

A_1 and A_2, are given. The case of interpolative fuzzy reasoning concerning two variables X and Y can be described through the modus ponens interpretation (13), as illustrated in Fig. 3

$$\begin{array}{l} \text{observation} : X \text{ is } A^* \\ \text{rules} : \text{ if } X \text{ is } A_1, \text{ then } Y \text{ is } B_1 \\ \qquad\quad \text{if } X \text{ is } A_2, \text{ then } Y \text{ is } B_2 \\ \hline \text{conclusion} : Y \text{ is } B^*? \end{array} \tag{13}$$

Here, $A_i = (a_{i0}, a_{i1}, a_{i2})$, $B_i = (b_{i0}, b_{i1}, b_{i2})$, $i = 1, 2$, and $A^* = (a_0, a_1, a_2)$, $B^* = (b_0, b_1, b_2)$.

To perform interpolation, the first step is to construct a new fuzzy set A' which has the same representative value as A^*. For this and by analogy to (8), the following is created first:

$$\begin{aligned} \lambda_{\text{Rep}} &= \frac{d(A_1, A^*)}{d(A_1, A_2)} \\ &= \frac{d(\text{Rep}(A_1), \text{Rep}(A^*))}{d(\text{Rep}(A_1), \text{Rep}(A_2))} \\ &= \frac{\frac{a_0+a_1+a_2}{3} - \frac{a_{10}+a_{11}+a_{12}}{3}}{\frac{a_{20}+a_{21}+a_{22}}{3} - \frac{a_{10}+a_{11}+a_{12}}{3}} \end{aligned} \tag{14}$$

where $d(A_1, A_2) = d(\text{Rep}(A_1), \text{Rep}(A_2))$ represents the distance between two fuzzy sets A_1 and A_2.

From this, a'_0, a'_1 and a'_2 of A' are calculated as follows:

$$a'_0 = (1 - \lambda_{\text{Rep}})a_{10} + \lambda_{\text{Rep}} a_{20} \tag{15}$$

$$a'_1 = (1 - \lambda_{\text{Rep}})a_{11} + \lambda_{\text{Rep}} a_{21} \tag{16}$$

$$a'_2 = (1 - \lambda_{\text{Rep}})a_{12} + \lambda_{\text{Rep}} a_{22} \tag{17}$$

which are collectively abbreviated to

$$A' = (1 - \lambda_{\text{Rep}})A_1 + \lambda_{\text{Rep}} A_2. \tag{18}$$

Now, A' has the same representative value as A^*.

Proof:

$$\text{Rep}(A') = \frac{a'_0 + a'_1 + a'_2}{3}.$$

With (15)–(17) and (14)

$$\begin{aligned} \text{Rep}(A') &= (1 - \lambda_{\text{Rep}})\frac{a_{10} + a_{11} + a_{12}}{3} \\ &\quad + \lambda_{\text{Rep}} \frac{a_{20} + a_{21} + a_{22}}{3} \\ &= (1 - \lambda_{\text{Rep}})\text{Rep}(A_1) + \lambda_{\text{Rep}}\text{Rep}(A_2) \\ &= \text{Rep}(A^*). \end{aligned}$$

■

Importantly, in so doing, A' is generated to be a convex fuzzy set as the following holds given $a_{10} \leq a_{11} \leq a_{12}$, $a_{20} \leq a_{21} \leq a_{22}$ and $0 \leq \lambda_{\text{Rep}} \leq 1$:

$$\begin{aligned} a'_1 - a'_0 &= (1 - \lambda_{\text{Rep}})(a_{11} - a_{10}) + \lambda_{\text{Rep}}(a_{21} - a_{20}) \geq 0 \\ a'_2 - a'_1 &= (1 - \lambda_{\text{Rep}})(a_{12} - a_{11}) + \lambda_{\text{Rep}}(a_{22} - a_{21}) \geq 0. \end{aligned}$$

The second step of performing interpolation is carried out in a similar way to the first, such that the consequent fuzzy set B' can be obtained as follows:

$$b'_0 = (1 - \lambda_{\text{Rep}})b_{10} + \lambda_{\text{Rep}} b_{20} \tag{19}$$

$$b'_1 = (1 - \lambda_{\text{Rep}})b_{11} + \lambda_{\text{Rep}} b_{21} \tag{20}$$

$$b'_2 = (1 - \lambda_{\text{Rep}})b_{12} + \lambda_{\text{Rep}} b_{22} \tag{21}$$

with abbreviated notation

$$B' = (1 - \lambda_{\text{Rep}})B_1 + \lambda_{\text{Rep}} B_2. \tag{22}$$

As a result, the newly derived rule $A' \Rightarrow B'$ involves the use of only normal and convex fuzzy sets.

As $A' \Rightarrow B'$ is derived from $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$, it is feasible to perform fuzzy reasoning with this new rule without further reference to its originals. The interpolative reasoning problem is therefore changed from expression (13) to the new *modus ponens* interpretation

$$\begin{array}{l} \text{observation} : X \text{ is } A^* \\ \text{rule} : \text{ if } X \text{ is } A', \text{ then } Y \text{ is } B' \\ \hline \text{conclusion} : Y \text{ is } B^*? \end{array} \tag{23}$$

This interpretation retains the same results as (13) in dealing with the extreme cases: If $A^* = A_1$, then it follows from (14) that $\lambda_{\text{Rep}} = 0$, and according to (18) and (22), $A' = A_1$ and $B' = B_1$, so the conclusion $B^* = B_1$. Similarly, if $A^* = A_2$, then $B^* = B_2$.

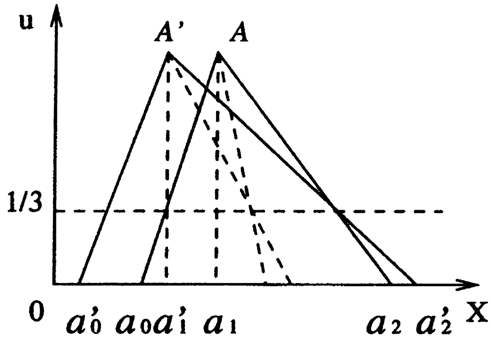

Fig. 4. Triangle scale transformation.

Other than the extreme cases, *similarity* measures are used to support the application of this new modus ponens as done in [11]. In particular, (23) can be interpreted as

$$\text{The more similar } X \text{ to } A', \text{ the more similar } Y \text{ to } B' \quad (24)$$

Suppose that a certain degree of similarity between A' and A^* is established, it is intuitive to require that the consequent parts B' and B^* attain the same similarity degree. The question is now how to obtain an operator which can represent the similarity degree between fuzzy sets A' and A^*, and to allow transforming B' to B^* with the desired degree of similarity. In this respect, two transformations are proposed as follows.

Scale Transformation: Given a *scale rate* s $(s \geq 0)$, in order to transform the current support $(a_2 - a_0)$, of fuzzy set $A = (a_0, a_1, a_2)$, into a new support $(s * (a_2 - a_0))$ while keeping the same representative value and ratio of left-support $(a'_1 - a'_0)$ to right-support $(a'_2 - a'_1)$ of the transformed fuzzy set, $A' = (a'_0, a'_1, a'_2)$, as those of its original, that is, $\mathrm{Rep}(A') = \mathrm{Rep}(A)$ and $((a'_1 - a'_0)/(a'_2 - a'_1)) = ((a_1 - a_0)/(a_2 - a_1))$, the new a'_0, a'_1 and a'_2 must satisfy (as illustrated in Fig. 4)

$$a'_0 = \frac{a_0(1+2s) + a_1(1-s) + a_2(1-s)}{3} \quad (25)$$
$$a'_1 = \frac{a_0(1-s) + a_1(1+2s) + a_2(1-s)}{3} \quad (26)$$
$$a'_2 = \frac{a_0(1-s) + a_1(1-s) + a_2(1+2s)}{3}. \quad (27)$$

In fact, to satisfy the conditions imposed over the transformation, the following linear equations must hold simultaneously:

$$\begin{cases} \frac{a'_0+a'_1+a'_2}{3} = \frac{a_0+a_1+a_2}{3} \\ \frac{a'_1-a'_0}{a'_2-a'_1} = \frac{a_1-a_0}{a_2-a_1} \\ a'_2 - a'_0 = s(a_2 - a_0) \end{cases}$$

Solving these equations leads to the solutions as given in (25)–(27). Note that this scale transformation guarantees that the transformed fuzzy sets are convex as the following holds given $a_0 \leq a_1 \leq a_2$ and $s \geq 0$:

$$a'_1 - a'_0 = s(a_1 - a_0) \geq 0$$
$$a'_2 - a'_1 = s(a_2 - a_1) \geq 0$$

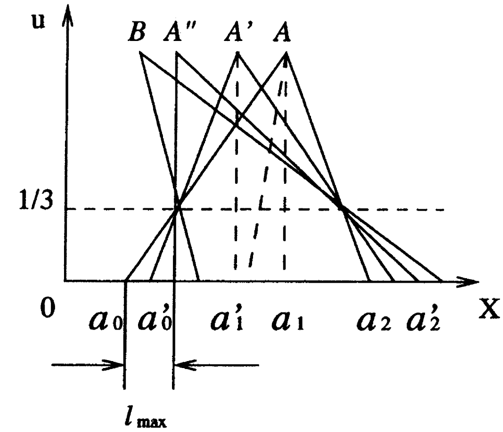

Fig. 5. Triangle move transformation.

The above shows how to obtain the resultant fuzzy set A' when the original fuzzy set A and a scale rate s are given. Conversely, in the case where two fuzzy sets $A = (a_0, a_1, a_2)$ and $A' = (a'_0, a'_1, a'_2)$ which have the same representative value are given, the scale rate is calculated as follows:

$$s = \frac{a'_2 - a'_0}{a_2 - a_0} \geq 0. \quad (28)$$

This measure reflects the similarity degree between A and A': the closer is s to 1, the more similar is A to A'. It is therefore used to act as, or to contribute to (see Section III-E for integrated transformation), the desirable similarity degree in order to transform B' to B^*.

Move Transformation: Given a moving distance l, in order to transform the current fuzzy support $(a_2 - a_0)$ from the starting location a_0 to a new starting position $a_0 + l$ while keeping the same representative value and length of support of the transformed fuzzy set as its original, i.e., $\mathrm{Rep}(A') = \mathrm{Rep}(A)$ and $a'_2 - a'_0 = a_2 - a_0$, the new a'_0, a'_1 and a'_2 must be (as shown in Fig. 5)

$$a'_0 = a_0 + l \quad (29)$$
$$a'_1 = a_1 - 2l \quad (30)$$
$$a'_2 = a_2 + l. \quad (31)$$

These can be obtained by solving the equations which are imposed to the transformation

$$\begin{cases} \frac{a'_0+a'_1+a'_2}{3} = \frac{a_0+a_1+a_2}{3} \\ a'_0 = a_0 + l \\ a'_2 - a'_0 = a_2 - a_0 \end{cases}$$

To ensure A' to be convex, the condition of $0 \leq l \leq l_{\max} = (a_1 - a_0)/3$ must hold. If $l > l_{\max}$, the transformation will generate nonconvex fuzzy sets. For instance, consider the extreme case in which A is transformed to A'', where the left slope of A'' becomes vertical (i.e. $a'_0 = a'_1$) as shown in Fig. 5. Here, $l = l_{\max}$. Any further increase in l will lead to the resulting transformed fuzzy set being a non-NCF set. To avoid this, the *move ratio* $\mathbb{M}$ is introduced

$$\mathbb{M} = \frac{l}{\frac{(a_1 - a_0)}{3}}. \tag{32}$$

The closer is $\mathbb{M}$ to 0, the less move (in terms of moving displacement l) is being made, and the closer is $\mathbb{M}$ to 1, the more move is being made. If move ratio $\mathbb{M} \in [0,1]$, then $l \leq l_{\max}$ holds. This ensures that the transformed fuzzy set A' is normal and convex if A is itself an NCF set.

Note that the move transformation has two possible moving directions, the above discusses the right-direction case (from the viewpoint of a_0) with $l \geq 0$, the left direction with $l \leq 0$ should hold by symmetry

$$\mathbb{M} = \frac{l}{\frac{(a_2 - a_1)}{3}} \in [-1, 0]. \tag{33}$$

As with the description for scale transformation, the above describes how to calculate resultant fuzzy set A' given the original fuzzy set A and a moving distance l (or move ratio $\mathbb{M}$). Now, consider the case where two convex triangular sets $A = (a_0, a_1, a_2)$ and $A' = (a'_0, a'_1, a'_2)$ which have the same representative value and the same support length are given, the move ratio $\mathbb{M}$ can be calculated as follows:

$$\mathbb{M} = \begin{cases} \frac{3(a'_0 - a_0)}{a_1 - a_0}, & if\ a'_0 \geq a_0 \\ \frac{3(a'_0 - a_0)}{a_2 - a_1}, & if\ a'_0 \leq a_0. \end{cases} \tag{34}$$

This reflects the similarity degree between A and A': the closer is $\mathbb{M}$ to 0, the more similar is A to A'. As A and A' are both convex, $\mathbb{M} \in [0,1]$ (when $a'_0 \geq a_0$) or $\mathbb{M} \in [-1, 0]$ (when $a'_0 \leq a_0$) must hold.

Thus, in general, the third step of the interpolation process is to calculate the similarity degree in terms of scale rate and move ratio between A' and A^*, and then obtain the resulting fuzzy set B^* by transforming B' with the same scale rate and move ratio.

Through interpolation steps 1–3, given a convex and normal triangular fuzzy set as the observation, a new convex and normal fuzzy set can be derived using two adjacent rules.

B. Single Antecedent Variable with Trapezoidal Fuzzy Sets

It is potentially very useful to extend the above interpolative reasoning method to applying to rules involving more complex fuzzy membership functions. This subsection describes the interpolation involving trapezoidal membership functions.

Consider a trapezoidal fuzzy set A, denoted by (a_0, a_1, a_2, a_3) for notation convenience, as shown in Fig. 6. The *bottom support, left slope, right slope* and *top support* of A are defined as $a_3 - a_0$, $a_1 - a_0$, $a_3 - a_2$, and $a_2 - a_1$, respectively. The representative value of A is defined as:

$$\mathrm{Rep}(A) = \frac{1}{3}\left(a_0 + \frac{a_1 + a_2}{2} + a_3\right). \tag{35}$$

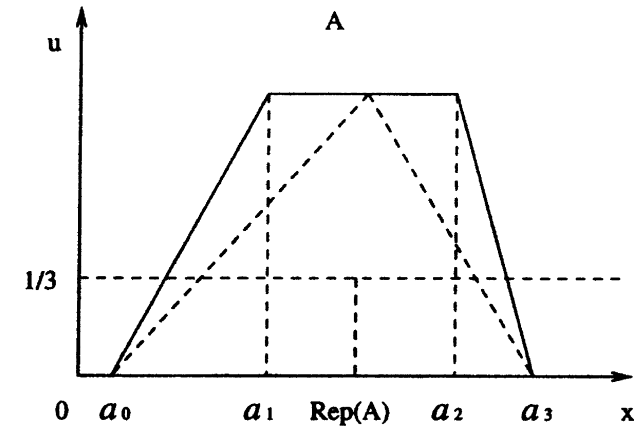

Fig. 6. Trapezoid representative value.

This definition subsumes the representative value of a triangular set as its specific case. This is because when a_1 and a_2 in a trapezoid are collapsed into a single value a_1, it degenerates into a triangle. In this case, the representative value definitions for trapezoidals (35) and triangles (12) remain the same. Of course, alternative definitions (e.g., $\mathrm{Rep}(A) = (a_0 + a_1 + a_2 + a_3)/4$) may be used to represent the overall location of a trapezoidal set, but this will destroy its compatibility to the triangular representation.

The calculation of the intermediate fuzzy rule $A' \Rightarrow B'$ follows a similar process as applying to triangular membership functions except that A' and B' here are trapezoidals rather than triangles. It is straightforward to verify the extreme cases (such as if $A^* = A_1$ then $B^* = B_1$) in the same way as with triangle cases. To adapt the proposed method to be suitable for trapezoidal fuzzy sets, attention is only drawn to the two transformations.

Scale Transformation: Given two *scale rates* s_b and s_t ($s_b \geq 0$ and $s_t \geq 0$) for bottom support scale and top support scale respectively, in order to transform the current bottom support $(a_3 - a_0)$ to the new bottom support $(s_b * (a_3 - a_0))$, and the top support $(a_2 - a_1)$ to the new top support $(s_t * (a_2 - a_1))$ while keeping the representative value and the ratio of left slope $(a'_1 - a'_0)$ to right slope $(a'_3 - a'_2)$ of the transformed fuzzy set the same as those of its original, that is, $\mathrm{Rep}(A') = \mathrm{Rep}(A)$ and $((a'_1 - a'_0)/(a'_3 - a'_2)) = ((a_1 - a_0)/(a_3 - a_2))$, the new a'_0, a'_1, a'_2 and a'_3 must satisfy (as illustrated in Fig. 7)

$$a'_0 = A - \frac{C(2a_1 + a_3 - 2a_0 - a_2) - D(a_1 + a_2 - a_0 - a_3)}{B} \tag{36}$$

$$a'_1 = A - \frac{C(a_0 + a_3 - a_1 - a_2) - D(5a_0 + a_2 - 5a_1 - a_3)}{B} \tag{37}$$

$$a'_2 = A - \frac{C(a_0 + a_3 - a_1 - a_2) - D(a_1 + 5a_3 - a_0 - 5a_2)}{B} \tag{38}$$

$$a'_3 = A - \frac{C(a_0 + 2a_2 - a_1 - 2a_3) - D(a_1 + a_2 - a_0 - a_3)}{B} \tag{39}$$

where $A = (2a_0 + a_1 + a_2 + 2a_3)/6$, $B = 6(a_1 + a_3 - a_0 - a_2)$, $C = 2s_b(a_3 - a_0)$ and $D = s_t(a_2 - a_1)$. These results can be

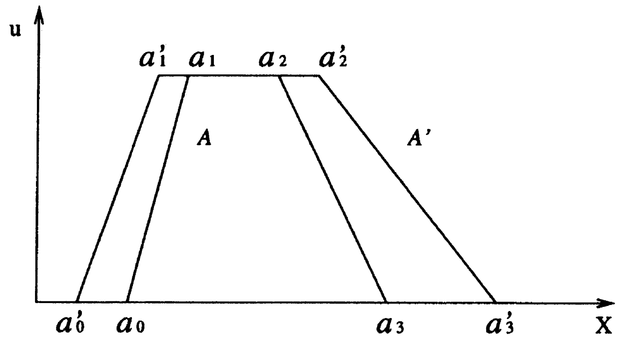

Fig. 7. Trapezoid scale transformation.

achieved by solving the following conditions, imposed over the transformation:

$$\begin{cases} \frac{1}{3}\left(a_0' + \frac{a_1'+a_2'}{2} + a_3'\right) = \frac{1}{3}\left(a_0 + \frac{a_1+a_2}{2} + a_3\right) \\ \frac{a_1'-a_0'}{a_3'-a_2'} = \frac{a_1-a_0}{a_3-a_2} \\ a_3' - a_0' = s_b(a_3 - a_0) \\ a_2' - a_1' = s_t(a_2 - a_1). \end{cases}$$

Note that the scale transformation guarantees that the transformed fuzzy sets are convex given that s_b and s_t ensure the bottom support of the resultant fuzzy set is wider than the top support and both left and right slopes are nonnegative. This can be shown by

$$\begin{aligned} a_1' - a_0' &= \frac{(a_1 - a_0)\left(\text{bot}(A') - \text{top}(A')\right)}{a_1 + a_3 - a_0 - a_2} \geq 0 \\ a_2' - a_1' &= s_t(a_2 - a_1) \geq 0 \\ a_3' - a_2' &= \frac{(a_3 - a_2)\left(\text{bot}(A') - \text{top}(A')\right)}{a_1 + a_3 - a_0 - a_2} \geq 0 \end{aligned}$$

where $\text{bot}(A')$ and $\text{top}(A')$ stand for the bottom and top supports' lengths of transformed fuzzy set A', respectively. However, arbitrarily choosing s_t when s_b is fixed may lead the top support of the resultant fuzzy set to becoming wider than the bottom support. To avoid this, the *scale ratio* $\mathbb{S}_t$, which represents the actual increase of the ratios between the top supports and the bottom supports, before and after the transformation, normalized over the maximal possible such increase (in the sense that it does not lead to nonconvexity), is introduced to restrict s_t with respect to s_b

$$\mathbb{S}_t = \begin{cases} \frac{\frac{s_t(a_2-a_1)}{s_b(a_3-a_0)} - \frac{a_2-a_1}{a_3-a_0}}{1-\frac{a_2-a_1}{a_3-a_0}}, & if\ s_t \geq s_b \geq 0 \\ \frac{\frac{s_t(a_2-a_1)}{s_b(a_3-a_0)} - \frac{a_2-a_1}{a_3-a_0}}{\frac{a_2-a_1}{a_3-a_0}}, & if\ s_b \geq s_t \geq 0, \end{cases} \tag{40}$$

Thus, if $\mathbb{S}_t \in [0,1]$ (when $s_t \geq s_b > 0$) or $\mathbb{S}_t \in [-1,0]$ (when $s_b \geq s_t > 0$), $s_b(a_3 - a_0) \geq s_t(a_2 - a_1)$, i.e., $\text{bot}(A') \geq \text{top}(A')$. This can be shown as follows.

Proof: When $s_t \geq s_b \geq 0$, assume $s_t(a_2 - a_1) > s_b(a_3 - a_0)$,

$$\begin{aligned} &\therefore\ \frac{s_t(a_2 - a_1)}{s_b(a_3 - a_0)} > 1. \\ &\because\ 1 \geq \frac{a_2 - a_1}{a_3 - a_0} \geq 0, \\ &\therefore\ \mathbb{S}_t > 1. \end{aligned}$$

This conflicts with $\mathbb{S}_t \in [0,1]$ and, hence, the assumption is wrong. So

$$s_b(a_3 - a_0) \geq s_t(a_2 - a_1).$$

When $s_b \geq s_t \geq 0$,

$$\begin{aligned} &\because\ a_3 - a_0 \geq a_2 - a_1, \\ &\therefore\ s_b(a_3 - a_0) \geq s_t(a_2 - a_1). \end{aligned}$$

■

If, however, two convex trapezoidal fuzzy sets $A = (a_0, a_1, a_2, a_3)$ and $A' = (a_0', a_1', a_2, a_3')$ happen to have the same representative value, the bottom scale rate of A, s_b, and the top scale ratio of A, $\mathbb{S}_t$, can be calculated as

$$s_b = \frac{a_3' - a_0'}{a_3 - a_0} \geq 0 \tag{41}$$

$$\mathbb{S}_t = \begin{cases} \frac{\frac{a_2'-a_1'}{a_3'-a_0'} - \frac{a_2-a_1}{a_3-a_0}}{1-\frac{a_2-a_1}{a_3-a_0}} \in [0,1], & if\ \frac{a_2'-a_1'}{a_2-a_1} \geq \frac{a_3'-a_0'}{a_3-a_0} \geq 0 \\ \frac{\frac{a_2'-a_1'}{a_3'-a_0'} - \frac{a_2-a_1}{a_3-a_0}}{\frac{a_2-a_1}{a_3-a_0}} \in [-1,0], & if\ \frac{a_3'-a_0'}{a_3-a_0} \geq \frac{a_2'-a_1'}{a_2-a_1} \geq 0 \end{cases} \tag{42}$$

Thus, in this case, s_b is free to take on any positive value while $\mathbb{S}_t \in [0,1]$ or $\mathbb{S}_t \in [-1,0]$ (depending on whether $((a_2' - a_1')/(a_2 - a_1)) \geq ((a_3' - a_0')/(a_3 - a_0))$ or not) must hold given that A and A' are both convex. The closer is $\mathbb{S}_t$ to 0, the closer is the ratio between $\text{top}(A')$ and $\text{bot}(A')$ to that between $\text{top}(A)$ and $\text{bot}(A)$. Correspondingly, the closer is $\mathbb{S}_t$ to 1, the closer is the ratio between $\text{top}(A')$ and $\text{bot}(A')$ to 1. Similarly, the closer is $\mathbb{S}_t$ to -1, the closer is the ratio between $\text{top}(A')$ and $\text{bot}(A')$ to 0. The ranges of $\mathbb{S}_t$ values are proven as follows.

Proof: When $((a_2' - a_1')/(a_2 - a_1)) \geq ((a_3' - a_0')/(a_3 - a_0)) \geq 0$,

$$\begin{aligned} &\because\ 1 \geq \frac{a_2' - a_1'}{a_3' - a_0'} \geq \frac{a_2 - a_1}{a_3 - a_0} \geq 0, \\ &\therefore\ 1 \geq \mathbb{S}_t \geq 0. \end{aligned}$$

When $((a_3' - a_0')/(a_3 - a_0)) \geq ((a_2' - a_1')/(a_2 - a_1)) \geq 0$

$$\begin{aligned} &\because\ 1 \geq \frac{a_2 - a_1}{a_3 - a_0} \geq \frac{a_2' - a_1'}{a_3' - a_0'} \geq 0, \\ &\therefore\ 0 \geq \mathbb{S}_t \geq -1. \end{aligned}$$

■

Move Transformation: Given a moving distance l, in order to transform the current fuzzy set from the starting location a_0 to a new starting position $a_0 + l$ while keeping the representative value, the length of bottom support $(a_3 - a_0)$ and the length of

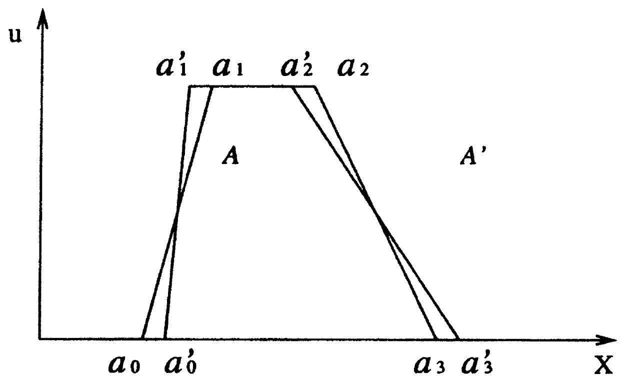

Fig. 8. Trapezoid move transformation.

the top support $(a_2 - a_1)$ all to remain the same, i.e., $\mathrm{Rep}(A') = \mathrm{Rep}(A)$, $a'_3 - a'_0 = a_3 - a_0$ and $a'_2 - a'_1 = a_2 - a_1$, the new a'_0, a'_1, a'_2 and a'_3 must be (as shown in Fig. 8):

$$a'_0 = a_0 + l \tag{43}$$
$$a'_1 = a_1 - 2l \tag{44}$$
$$a'_2 = a_2 - 2l \tag{45}$$
$$a'_3 = a_3 + l \tag{46}$$

These can be obtained by solving the equations which are imposed to the transformation:

$$\begin{cases} \frac{1}{3}\left(a'_0 + \frac{a'_1 + a'_2}{2} + a'_3\right) = \frac{1}{3}\left(a_0 + \frac{a_1 + a_2}{2} + a_3\right) \\ a'_0 = a_0 + l \\ a'_3 - a'_0 = a_3 - a_0 \\ a'_2 - a'_1 = a_2 - a_1 \end{cases}$$

To ensure A' to be convex, the condition of $0 \le l \le l_{\max} = (a_1 - a_0)/3$ must hold. If $l > l_{\max}$, the transformation will generate nonconvex fuzzy sets. As with triangular case, the *move ratio* $\mathbb{M}$ is introduced to avoid nonconvexity:

$$\mathbb{M} = \frac{l}{\frac{(a_1 - a_0)}{3}}. \tag{47}$$

If the move ratio $\mathbb{M} \in [0, 1]$, then $l \le l_{\max}$ holds. Similar to triangle move transformation, there is another moving direction with $l \le 0$. In that case the condition

$$\mathbb{M} = \frac{l}{\frac{(a_3 - a_2)}{3}} \in [-1, 0] \tag{48}$$

is imposed to ensure the convexity of the transformed fuzzy sets.

As with the scale transformation, if two convex trapezoidal sets $A = (a_0, a_1, a_2, a_3)$ and $A' = (a'_0, a'_1, a'_2, a'_3)$ which have the same representative value and the same support lengths are given, the move ratio $\mathbb{M}$ can be calculated as follows:

$$\mathbb{M} = \begin{cases} \frac{3(a'_0 - a_0)}{a_1 - a_0} & if\ a'_0 \ge a_0 \\ \frac{3(a'_0 - a_0)}{a_3 - a_2} & if\ a'_0 \le a_0 \end{cases} \tag{49}$$

As A and A' are both convex, $\mathbb{M} \in [0, 1]$ if $a'_0 \ge a_0$ or $\mathbb{M} \in [-1, 0]$ if $a'_0 \le a_0$.

It is easy to see that triangular transformation is a specific case of trapezoidal transformation. In fact, if $a_1 = a_2$ the trapezoid becomes a triangle. Substituting $a_1 = a_2$ and $s_t = 0$ in trapezoidal transformation formulae (36)–(39) and (43)–(46) leads to the same results as triangular transformation formulae (25)–(27) and (29)–(31).

C. Single Antecedent Variable with Hexagonal Fuzzy Sets

A fairly general case, the interpolation of the hexagonal fuzzy sets, is described in this subsection. This is to be followed by a straightforward extension in order to deal with the interpolation of any complex polygonal fuzzy membership functions later. One open issue for such an extension is to determine the representative value for a given complex, asymmetrical polygonal fuzzy set. For computational simplicity, the average of the x coordinate values of all odd points is defined as the representative value for any more complex polygon than trapezoidals.

Consider a generalized hexagonal fuzzy set A, denoted as $(a_0, a_1, a_2, a_3, a_4, a_5)$, as shown in Fig. 9, a_2 and a_3 are the two normal, odd points (whose membership values are 1), a_0 and a_5 are the two extreme, odd points (whose membership values are 0), and a_1 and a_4 are the two intermediate, odd points (whose membership values are the same and both are between 0 and 1 exclusively). For notational convenience, three *supports* (the horizontal intervals between a pair of odd points which involve the same membership value) are denoted as the *bottom support* $(a_5 - a_0)$, *middle support* $(a_4 - a_1)$ and *top support* $(a_3 - a_2)$, and four *slopes* (nonhorizontal intervals between two consecutive odd points) are denoted as $a_1 - a_0$, $a_2 - a_1$, $a_4 - a_3$ and $a_5 - a_4$. Also, as indicated before, for computational simplicity, the *average representative value* of A is defined as

$$\mathrm{Rep}(A) = \frac{a_0 + a_1 + a_2 + a_3 + a_4 + a_5}{6}. \tag{50}$$

Note that alternative definitions may be used to apply the transformations. For example, the *compatible representative value*, which is compatible to the less complex fuzzy sets (including triangular, trapezoidal and pentagonal fuzzy sets), can be defined as:

$$\mathrm{Rep}(A) = \frac{1}{3}\left[a_0 + \left(1 - \frac{\alpha}{2}\right)(a_1 - a'_1) + \frac{1}{2}(a_2 + a_3) + \left(1 - \frac{\alpha}{2}\right)(a_4 - a'_4) + a_5\right] \tag{51}$$

where $a'_1 = \alpha a_2 + (1 - \alpha)a_0$ and $a'_4 = \alpha a_3 + (1 - \alpha)a_5$ (see Fig. 9). Another alternative definition, *weighted average representative value* makes use of fuzzy membership values:

$$\mathrm{Rep}(A) = \frac{1}{4+\alpha}\left[\frac{1}{2}(a_0 + a_5) + \frac{(1+\alpha)}{2}(a_1 + a_4) + a_2 + a_3\right] \tag{52}$$

where α is the membership value of both a_1 and a_4. This definition assumes that the weights (from 1/2 to 1) assigned to points increase upwards from the bottom support to the top support, to reflect the significance of the fuzzy membership values. The weighted average of the odd points is then taken as the representative value of such a fuzzy set. Of course, the range of the weights ([1/2,1]) is optional. One of the most widely used defuzzification methods—the center of core can also be used to

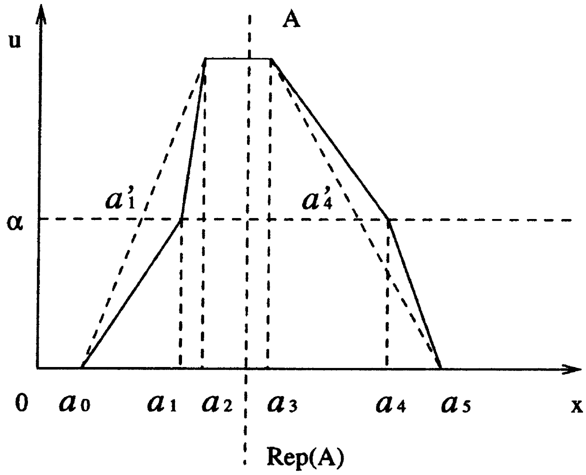

Fig. 9. Hexagon representative value.

define the *center of core representative value*. In this case, the representative value is solely determined by those points with a fuzzy membership value of 1:

$$\text{Rep}(A) = \frac{1}{2}(a_2 + a_3). \tag{53}$$

The interpolation by using either of these alternative definitions follows the same procedure as the one employing simple definition (50).

The calculation of intermediate fuzzy rule $A' \Rightarrow B'$ follows the triangular and/or trapezoidal cases. The attention is again drawn to the scale and move transformations as described as follows.

Scale Transformation: Given three *scale rates* s_b, s_m and s_t ($s_b \geq 0$, $s_m \geq 0$ and $s_t \geq 0$) representing the scales applied to bottom support, middle support and top support, respectively, the fuzzy set $A = (a_0, a_1, a_2, a_3, a_4, a_5)$ can be transformed to $A' = (a_0', a_1', a_2', a_3', a_4', a_5')$ by solving

$$\begin{cases} \frac{a_0'+a_1'+a_2'+a_3'+a_4'+a_5'}{6} = \frac{a_0+a_1+a_2+a_3+a_4+a_5}{6} \\ \frac{a_1'-a_0'}{a_5'-a_4'} = \frac{a_1-a_0}{a_5-a_4} \\ \frac{a_2'-a_1'}{a_4'-a_3'} = \frac{a_2-a_1}{a_4-a_3} \\ a_5' - a_0' = s_b(a_5 - a_0) \\ a_4' - a_1' = s_m(a_4 - a_1) \\ a_3' - a_2' = s_t(a_3 - a_2). \end{cases}$$

The solution of this is omitted here. As with the trapezoidal case, the resultant fuzzy set A' must be of the property that $a_0' \leq a_1' \leq a_2' \leq a_3' \leq a_4' \leq a_5'$, given that the desired top support is narrower than the middle support and the middle support is narrower than the bottom support. Therefore, certain constraints should be imposed over s_m if s_b is fixed, and over s_t if s_m is fixed. For this reason, the scale ratios of middle and top supports of A, denoted as $\mathbb{S}_m$ and $\mathbb{S}_t$, are introduced to constrain the scale rates s_m and s_t respectively:

$$\mathbb{S}_m = \begin{cases} \frac{\frac{s_m(a_4-a_1)}{s_b(a_5-a_0)} - \frac{a_4-a_1}{a_5-a_0}}{1-\frac{a_4-a_1}{a_5-a_0}} & if\ s_m \geq s_b \geq 0 \\ \frac{\frac{s_m(a_4-a_1)}{s_b(a_5-a_0)} - \frac{a_4-a_1}{a_5-a_0}}{\frac{a_4-a_1}{a_5-a_0}} & if\ s_b \geq s_m \geq 0 \end{cases} \tag{54}$$

$$\mathbb{S}_t = \begin{cases} \frac{\frac{s_t(a_3-a_2)}{s_m(a_4-a_1)} - \frac{a_3-a_2}{a_4-a_1}}{1-\frac{a_3-a_2}{a_4-a_1}} & if\ s_t \geq s_m \geq 0 \\ \frac{\frac{s_t(a_3-a_2)}{s_m(a_4-a_1)} - \frac{a_3-a_2}{a_4-a_1}}{\frac{a_3-a_2}{a_4-a_1}} & if\ s_m \geq s_t \geq 0. \end{cases} \tag{55}$$

If $\mathbb{S}_m \in [0, 1]$ (when $s_m \geq s_b \geq 0$) or $\mathbb{S}_m \in [-1, 0]$ (when $s_b \geq s_m \geq 0$) and $\mathbb{S}_t \in [0, 1]$ (when $s_t \geq s_m \geq 0$) or $\mathbb{S}_t \in [-1, 0]$ (when $s_m \geq s_t \geq 0$), then $a_5' - a_0' \geq a_4' - a_1' \geq a_3' - a_2'$. The proof is referred to Section III-D for the general polygonal fuzzy membership function case. The constraints of $\mathbb{S}_m$ and $\mathbb{S}_t$ along with the scale transformation thus lead to a unique, normal and convex fuzzy set A'.

Conversely, if two convex hexagonal fuzzy sets $A = (a_0, a_1, a_2, a_3, a_4, a_5)$ and $A' = (a_0', a_1', a_2', a_3', a_4', a_5')$ which have the same representative value are given, the scale rate of the bottom support, s_b, and the scale ratios of the middle and top supports, $\mathbb{S}_m$ and $\mathbb{S}_t$, are calculated as

$$s_b = \frac{a_5' - a_0'}{a_5 - a_0} \geq 0 \tag{56}$$

$$\mathbb{S}_m = \begin{cases} \frac{\frac{a_4'-a_1'}{a_5'-a_0'} - \frac{a_4-a_1}{a_5-a_0}}{1-\frac{a_4-a_1}{a_5-a_0}} \in [0,1] & if\ \frac{a_4'-a_1'}{a_4-a_1} \geq \frac{a_5'-a_0'}{a_5-a_0} \geq 0 \\ \frac{\frac{a_4'-a_1'}{a_5'-a_0'} - \frac{a_4-a_1}{a_5-a_0}}{\frac{a_4-a_1}{a_5-a_0}} \in [-1,0], & if\ \frac{a_5'-a_0'}{a_5-a_0} \geq \frac{a_4'-a_1'}{a_4-a_1} \geq 0 \end{cases} \tag{57}$$

$$\mathbb{S}_t = \begin{cases} \frac{\frac{a_3'-a_2'}{a_4'-a_1'} - \frac{a_3-a_2}{a_4-a_1}}{1-\frac{a_3-a_2}{a_4-a_1}} \in [0,1], & if\ \frac{a_3'-a_2'}{a_3-a_2} \geq \frac{a_4'-a_1'}{a_4-a_1} \geq 0 \\ \frac{\frac{a_3'-a_2'}{a_4'-a_1'} - \frac{a_3-a_2}{a_4-a_1}}{\frac{a_3-a_2}{a_4-a_1}} \in [-1,0], & if\ \frac{a_4'-a_1'}{a_4-a_1} \geq \frac{a_3'-a_2'}{a_3-a_2} \geq 0 \end{cases} \tag{58}$$

Again, the proof of $\mathbb{S}_m \in [-1, 1]$ and $\mathbb{S}_t \in [-1, 1]$ given that A and A' both are convex is referred to Section III-D.

Move Transformation: It is slightly more complicated to apply move transformation to hexagonal fuzzy sets although it still follows the same principle. Compared to the cases of triangular and trapezoidal fuzzy sets, where only one move transformation is carried out in order to obtain the resultant fuzzy set, this case needs two moves (referred to as sub-moves hereafter) to achieve the resultant fuzzy set.

Given two moving distances l_b and l_m, in order to transform the bottom support of the fuzzy set $A = (a_0, a_1, a_2, a_3, a_4, a_5)$ from the starting location a_0 to a new starting position $a_0' = a_0 + l_b$, and to transform the middle support from a_1 to $a_1 = a_1 + l_m$ while keeping the representative value and the lengths of three supports to remain the same (as shown in Fig. 10), two sub-moves are carried out.

First, a sub-move to the desired bottom support position is attempted. If it is to be moved to the right direction from a_0's point of view, $0 \leq l_b \leq l_{b\max} = (((a_0 + a_1 + a_2)/3) - a_0)$ must hold. In the extreme position where $l_b = l_{b\max}$, the resultant fuzzy set $A'' = (a_0'', a_1'', a_2'', a_3'', a_4'', a_5'')$, i.e., the dotted hexagonal set in Fig. 10, has $a_0'' = a_1'' = a_2''$. If $l_b > l_{b\max}$, it will lead to a nonconvex fuzzy set. As with the triangular and trapezoidal cases, the bottom move ratio is introduced to avoid this potential nonconvexity

$$\mathbb{M}_b = \frac{l_b}{\frac{a_0+a_1+a_2}{3} - a_0}. \tag{59}$$

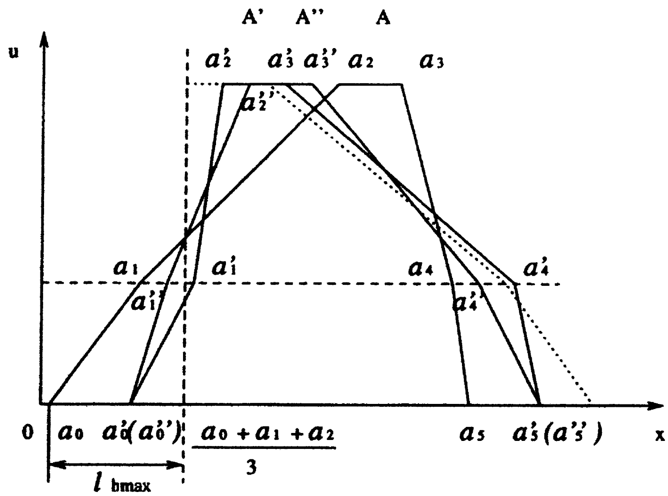

Fig. 10. Hexagon bottom move transformation.

If the move ratio $\mathbb{M}_b \in [0,1]$, then $l_b \leq l_{\max}$ holds. The moving distance of the point a_i $(i = 0,1,2)$ is calculated by multiplying $\mathbb{M}_b$ with the distance between the extreme position $((a_0 + a_1 + a_2)/3)$ and itself. In so doing, a_0, a_1, and a_2 will move the same proportion of their respective distance to the extreme position. The other three points a_3, a_4, and a_5 can therefore be determined by attaining the same lengths of the three supports, respectively. The fuzzy set A' after this sub-move is thus calculated by

$$a_0'' = a_0 + \mathbb{M}_b \left(\frac{a_0 + a_1 + a_2}{3} - a_0 \right) \tag{60}$$

$$a_1'' = a_1 + \mathbb{M}_b \left(\frac{a_0 + a_1 + a_2}{3} - a_1 \right) \tag{61}$$

$$a_2'' = a_2 + \mathbb{M}_b \left(\frac{a_0 + a_1 + a_2}{3} - a_2 \right) \tag{62}$$

$$a_3'' = a_3 + \mathbb{M}_b \left(\frac{a_0 + a_1 + a_2}{3} - a_2 \right) \tag{63}$$

$$a_4'' = a_4 + \mathbb{M}_b \left(\frac{a_0 + a_1 + a_2}{3} - a_1 \right) \tag{64}$$

$$a_5'' = a_5 + \mathbb{M}_b \left(\frac{a_0 + a_1 + a_2}{3} - a_0 \right). \tag{65}$$

From (60)–(65), it is clear that A'' is convex as the following holds given $\mathbb{M}_b \in [0,1]$:

$$\begin{aligned} a_1'' - a_0'' &= (a_1 - a_0)(1 - \mathbb{M}_b) \geq 0 \\ a_2'' - a_1'' &= (a_2 - a_1)(1 - \mathbb{M}_b) \geq 0 \\ a_3'' - a_2'' &= a_3 - a_2 \geq 0 \\ a_4'' - a_3'' &= a_4 - a_3 + \mathbb{M}_b(a_2 - a_1) \geq 0 \\ a_5'' - a_4'' &= a_5 - a_4 + \mathbb{M}_b(a_1 - a_0) \geq 0. \end{aligned}$$

It can be verified that A'' has the same representative value as A. This is because, according to (60)–(65),

$$\begin{aligned} \mathrm{Rep}(A'') &= \frac{a_0'' + a_1'' + a_2'' + a_3'' + a_4'' + a_5''}{6} \\ &= \frac{a_0 + a_1 + a_2 + a_3 + a_4 + a_5}{6} \\ &= \mathrm{Rep}(A). \end{aligned}$$

For the opposite moving direction where $l_b \leq 0$, the condition

$$\mathbb{M}_b = \frac{l_b}{a_5 - \frac{a_3 + a_4 + a_5}{3}} \in [-1, 0] \tag{66}$$

is imposed to ensure the convexity of the transformed fuzzy set. The results of A'' can be similarly written as

$$a_0'' = a_0 + \mathbb{M}_b \left(a_5 - \frac{a_3 + a_4 + a_5}{3} \right) \tag{67}$$

$$a_1'' = a_1 + \mathbb{M}_b \left(a_4 - \frac{a_3 + a_4 + a_5}{3} \right) \tag{68}$$

$$a_2'' = a_2 + \mathbb{M}_b \left(a_3 - \frac{a_3 + a_4 + a_5}{3} \right) \tag{69}$$

$$a_3'' = a_3 + \mathbb{M}_b \left(a_3 - \frac{a_3 + a_4 + a_5}{3} \right) \tag{70}$$

$$a_4'' = a_4 + \mathbb{M}_b \left(a_4 - \frac{a_3 + a_4 + a_5}{3} \right) \tag{71}$$

$$a_5'' = a_5 + \mathbb{M}_b \left(a_5 - \frac{a_3 + a_4 + a_5}{3} \right). \tag{72}$$

Of course, it can be proved from (67)–(72) that this resultant fuzzy set is indeed convex given $\mathbb{M}_b \in [-1, 0]$

$$\begin{aligned} a_1'' - a_0'' &= a_1 - a_0 + \mathbb{M}_b(a_4 - a_5) \geq 0 \\ a_2'' - a_1'' &= a_2 - a_1 + \mathbb{M}_b(a_3 - a_4) \geq 0 \\ a_3'' - a_2'' &= a_3 - a_2 \geq 0 \\ a_4'' - a_3'' &= (a_4 - a_3)(1 + \mathbb{M}_b) \geq 0 \\ a_5'' - a_4'' &= (a_5 - a_4)(1 + \mathbb{M}_b) \geq 0. \end{aligned}$$

Again, A'' and A have the same representative value, ensured by (67)–(72).

In both cases ($l_b \geq 0$ and $l_b \leq 0$), $a_0'' = a_0 + l_b$ holds. This means the bottom support of A is moved to the desired place after the first sub-move. So the second sub-move is aimed to move the middle and the top supports to the desired places from A'' to A' as shown in Fig. 10. This sub-move does not affect the place of the bottom support as it has already been in the right place. Thus this step is almost the same as the move proposed for a trapezoidal fuzzy set except that the maximal moving distance (in the sense that it does not lead to nonconvexity) should be less than, or at most equal to $(a_2'' - a_1'')/2$ (not $(a_2'' - a_1'')/3$ as in trapezoidal case due to the difference in the representative definition for hexagonal fuzzy sets), if moving the middle support to the right direction (i.e., the new move displacement $l_m' = l_m - (a_1'' - a_1) \geq 0$). This is because the maximal moving distance is also constrained to the bottom support (i.e., $l_m' \leq a_5'' - a_4''$) as it may move a_4'' to a place exceeding a_5''. It is intuitive to pick the minimal values of the two distances as the maximal moving distance. The move ratio can, therefore, be defined as

$$\mathbb{M}_m = \frac{l_m - (a_1'' - a_1)}{min\left\{ \frac{a_2'' - a_1''}{2}, a_5'' - a_4'' \right\}}. \tag{73}$$

When applying the second sub-move, consider that both above and below nonconvexity may arise, the *applied move ratio* $\mathbb{M}'_m$ is introduced as

$$\mathbb{M}'_m = \mathbb{M}_m \frac{\min\left\{\frac{a''_2 - a''_1}{2}, a''_5 - a''_4\right\}}{\frac{a''_2 - a''_1}{2}}. \tag{74}$$

If $\mathbb{M}_m \in [0,1]$, $\mathbb{M}'_m \in [0, \mathbb{M}_m]$. The introduction of applied move ratio avoids the potential below nonconvexity when applying sub-move as follows:

$$a'_0 = a''_0 \tag{75}$$

$$a'_1 = a''_1 + \mathbb{M}'_m\left(\frac{a''_1 + a''_2}{2} - a''_1\right) \tag{76}$$

$$a'_2 = a''_2 + \mathbb{M}'_m\left(\frac{a''_1 + a''_2}{2} - a''_2\right) \tag{77}$$

$$a'_1 = a''_3 + \mathbb{M}'_m\left(\frac{a''_1 + a''_2}{2} - a''_2\right), \tag{78}$$

$$a'_1 = a''_4 + \mathbb{M}'_m\left(\frac{a''_1 + a''_2}{2} - a''_1\right) \tag{79}$$

$$a'_5 = a''_5. \tag{80}$$

Merging (73) and (74) into (76) and (77) leads to $a'_1 = a_1 + l_m$ and $a'_2 = a_2 - l_b - l_m$, which are the desired positions for a_1 and a_2 to be moved on to, respectively. It can also be shown that A' is an NCF set and $\mathrm{Rep}(A') = \mathrm{Rep}(A'') = \mathrm{Rep}(A)$. All these properties are maintained if on the opposite case where $l'_m \leq 0$.

As discussed above, if given two move ratios $\mathbb{M}_b \in [-1,1]$ and $\mathbb{M}_m \in [-1,1]$, the two sub-moves transform the given NCF set $A = (a_0, a_1, a_2, a_3, a_4, a_5)$ to a new NCF set $A' = (a'_0, a'_1, a'_2, a'_3, a'_4, a'_5)$ while keeping the representative values and the lengths of supports to be the same.

Conversely, if two convex hexagonal fuzzy sets $A = (a_0, a_1, a_2, a_3, a_4, a_5)$ and $A' = (a'_0, a'_1, a'_2, a'_3, a'_4, a'_5)$ are given, which have the same representative value and the same support lengths, the move ratios which are calculated in an order from bottom to top must lie between $[-1,1]$. First, the bottom move ratio is computed by

$$\mathbb{M}_b = \begin{cases} \frac{a'_0 - a_0}{\frac{a_0 + a_1 + a_2}{3} - a_0} \in [0,1] & if\ a'_0 \geq a_0 \\ \frac{a'_0 - a_0}{a_5 - \frac{a_3 + a_4 + a_5}{3}} \in [-1,0] & if\ a'_0 \leq a_0 \end{cases} \tag{81}$$

It is used to carry out the first sub-move of A to generate $A'' = (a''_0, a''_1, a''_3, a''_4, a''_5)$ according to (60)–(65) or (67)–(72). Then, the middle move ratio can be calculated by:

$$\mathbb{M}_m = \begin{cases} \frac{a'_1 - a''_1}{\min\left\{\frac{a''_2 - a''_1}{2}, a''_5 - a''_4\right\}} \in [0,1], & if\ a'_1 \geq a''_1 \\ \frac{a'_1 - a''_1}{\min\left\{\frac{a''_4 - a''_3}{2}, a''_1 - a''_0\right\}} \in [-1,0], & if a'_1 \leq a''_1. \end{cases} \tag{82}$$

The proof of the ranges of $\mathbb{M}_b$ and $\mathbb{M}_m$ is omitted here.

D. Single Antecedent Variable With More Complex Fuzzy Sets

Any complex polygonal fuzzy sets can be similarly dealt with by following an analogous procedure to the hexagonal fuzzy sets. Fig. 11 shows the whole range of polygonal membership sets from the triangular to the polygonal function with arbitrary n odd points. Clearly, a general fuzzy membership function with n odd points has $\lfloor n/2 \rfloor$ supports (horizontal intervals between a pair of odd points which have the same membership value) and $2(\lceil n/2 \rceil - 1)$ slopes (nonhorizontal intervals between two consecutive odd points). Several specific cases are summarized in Table I.

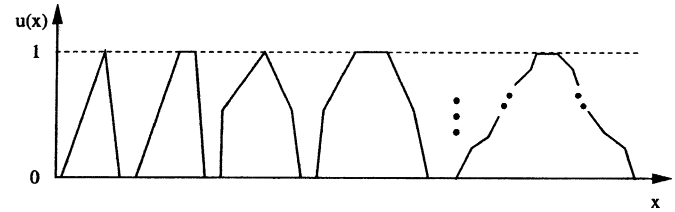

Fig. 11. Polygonal fuzzy membership functions.

TABLE I
VALUES OF $\lfloor n/2 \rfloor$, $2\lceil n/2 \rceil - 2$ AND $\lceil n/2 \rceil - 1$ GIVEN n

n	3	4	5	6
$\lfloor \frac{n}{2} \rfloor$	1	2	2	3
$2(\lceil \frac{n}{2} \rceil - 1)$	2	2	4	4
$\lceil \frac{n}{2} \rceil - 1$	1	1	2	2

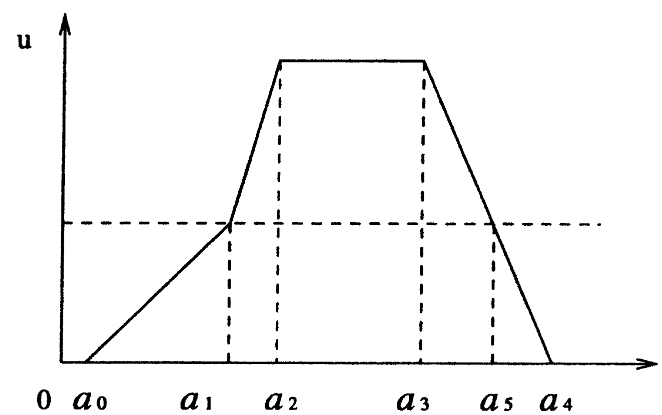

Fig. 12. A polygonal fuzzy membership function without evenly paired odd points.

However, a given polygonal fuzzy set does not need to have such evenly paired odd points. For example, a fuzzy membership function with 5 odd points (as shown in Fig. 12) has two pairs of odd points a_0 and a_4, and a_2 and a_3, but lack an odd point to form a pair with a_1. In this case, an additional point a_5 is artificially created so that the fuzzy set can still have pairs of "odd" points. Without losing generality, it is therefore permitted to assume that any given polygonal set can be represented by evenly paired odd points (with or without artificially created additional points).

Consider applying scale transformation to an arbitrary polygonal fuzzy membership function $A = (a_0, \ldots, a_{n-1})$ (as shown in Fig. 13) to generate $A' = (a'_0, \ldots, a'_{n-1})$ such that they have the same representative value, and $a'_{n-1-i} - a'_i = s_i(a_{n-1-i} - a_i)$, $i = \{0, \ldots, \lfloor n/2 \rfloor - 1\}$. In order to achieve this, $\lfloor n/2 \rfloor$ equations $a'_{n-1-i} - a'_i = s_i(a_{n-1-i} - a_i), i = \{0, \ldots, \lfloor n/2 \rfloor - 1\}$, are imposed to obtain the supports with desired lengths, and $(\lceil n/2 \rceil - 1)$ equations $((a'_{i+1} - a'_i)/(a'_{n-1-i} - a'_{n-2-i}))= ((a_{i+1} - a_i)/(a_{n-1-i} - a_{n-2-i}))$, $i = \{0, \ldots, \lceil n/2 \rceil - 2\}$

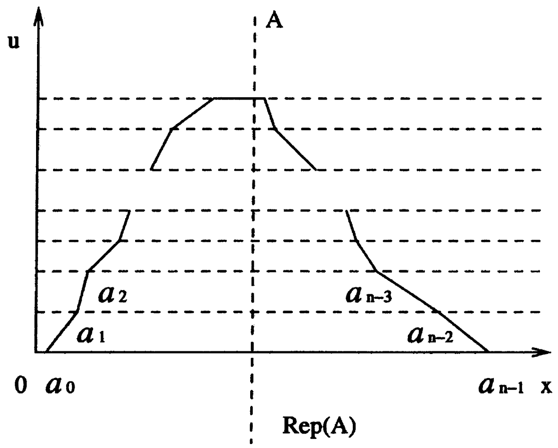

Fig. 13. Arbitrary polygonal fuzzy membership function with n odd points.

are imposed to equalise the ratios between the left $(\lceil n/2\rceil - 1)$ slopes' lengths and the right $(\lceil n/2\rceil - 1)$ slopes' lengths of A' to those counterparts of the original fuzzy set A. The equation $(a_0 + \ldots + a_{n-1})/n = (a'_0 + \ldots + a'_{n-1})/n$ which ensures the same representative values before and after the transformation is added to make up of $\lfloor n/2\rfloor + (\lceil n/2\rceil - 1) + 1 = n$ equations. Solving these n equations simultaneously results in an unique and convex fuzzy set A' given that the resultant set has the descending order of the support lengths from the bottom to the top. This can be guaranteed if the scale ratio of ith support $(i = \{1, 2, \ldots, \lfloor n/2\rfloor - 1\})$, denoted as $\mathbb{S}_i$, lies in the range [0,1] or [−1,0] (depending on whether $s_i \geq s_{i-1}$ or not). It can be mathematically expressed as:

$$\mathbb{S}_i = \begin{cases} \dfrac{\frac{s_i(a_{n-i-1}-a_i)}{s_{i-1}(a_{n-i}-a_{i-1})} - \frac{a_{n-i-1}-a_i}{a_{n-i}-a_{i-1}}}{1 - \frac{a_{n-i-1}-a_i}{a_{n-i}-a_{i-1}}} \in [0,1], & if\ s_i \geq s_{i-1} \geq 0 \\ \dfrac{\frac{s_i(a_{n-i-1}-a_i)}{s_{i-1}(a_{n-i}-a_{i-1})} - \frac{a_{n-i-1}-a_i}{a_{n-i}-a_{i-1}}}{\frac{a_{n-i-1}-a_i}{a_{n-i}-a_{i-1}}} \in [-1,0], & if\ s_{i-1} \geq s_i \geq 0. \end{cases} \tag{83}$$

Proof: When $s_i \geq s_{i-1} \geq 0$, assume $s_i(a_{n-i-1} - a_i) > s_{i-1}(a_{n-i} - a_{i-1})$

$$\therefore\ \frac{s_i(a_{n-i-1} - a_i)}{s_{i-1}(a_{n-i} - a_{i-1})} > 1.$$

Also

$$\because\ 1 \geq \frac{a_{n-i-1} - a_i}{a_{n-i} - a_{i-1}} \geq 0,$$
$$\therefore\ \mathbb{S}_i > 1.$$

This conflicts with $\mathbb{S}_i \in [0, 1]$. The assumption is therefore wrong. So $s_{i-1}(a_{n-i} - a_{i-1}) \geq s_i(a_{n-i-1} - a_i)$.

When $s_{i-1} \geq s_i \geq 0$

$$\because\ a_{n-i} - a_{i-1} \geq a_{n-i-1} - a_i$$
$$\therefore\ s_{i-1}(a_{n-i} - a_{i-1}) \geq s_i(a_{n-i-1} - a_i).$$

■

Conversely, if two convex sets $A = (a_0, \ldots, a_{n-1})$ and $A' = (a'_0, \ldots, a'_{n-1})$ which have the same representative value are given, the scale rate of the bottom support, s_0, and the scale ratio of the i-th support, $\mathbb{S}_i$ ($\mathbb{S}_i, i = \{1, \ldots, \lfloor n/2\rfloor - 1\}$) can be calculated by

$$s_0 = \frac{a'_{n-1} - a'_0}{a_{n-1} - a_0} \tag{84}$$

$$\mathbb{S}_i = \begin{cases} \dfrac{\frac{a'_{n-i-1}-a'_i}{a'_{n-i}-a'_{i-1}} - \frac{a_{n-i-1}-a_i}{a_{n-i}-a_{i-1}}}{1 - \frac{a_{n-i-1}-a_i}{a_{n-i}-a_{i-1}}} \in [0,1] \\ \left(if\ \frac{a'_{n-i-1}-a'_i}{a_{n-i-1}-a_i} \geq \frac{a'_{n-i}-a'_{i-1}}{a_{n-i}-a_{i-1}} \geq 0\right) \\ \dfrac{\frac{a'_{n-i-1}-a'_i}{a'_{n-i}-a'_{i-1}} - \frac{a_{n-i-1}-a_i}{a_{n-i}-a_{i-1}}}{\frac{a_{n-i-1}-a_i}{a_{n-i}-a_{i-1}}} \in [-1,0] \\ \left(if\ \frac{a'_{n-i}-a'_{i-1}}{a_{n-i}-a_{i-1}} \geq \frac{a'_{n-i-1}-a'_i}{a_{n-i-1}-a_i} \geq 0\right). \end{cases} \tag{85}$$

Given that A and A' both are convex, the ranges of $\mathbb{S}_i$ can be proved as follows.

Proof: When $((a'_{n-i-1} - a'_i)/(a_{n-i-1} - a_i)) \geq ((a'_{n-i} - a'_{i-1})/(a_{n-i} - a_{i-1})) \geq 0$

$$\because\ 1 \geq \frac{a'_{n-i-1} - a'_i}{a'_{n-i} - a'_{i-1}} \geq \frac{a_{n-i-1} - a_i}{a_{n-i} - a_{i-1}} \geq 0$$
$$\therefore\ 1 \geq \mathbb{S}_i \geq 0.$$

When $((a'_{n-i} - a'_{i-1})/(a_{n-i} - a_{i-1})) \geq ((a'_{n-i-1} - a'_i)/(a_{n-i-1} - a_i)) \geq 0$

$$\because\ 1 \geq \frac{a_{n-i-1} - a_i}{a_{n-i} - a_{i-1}} \geq \frac{a'_{n-i-1} - a'_i}{a'_{n-i} - a'_{i-1}} \geq 0$$
$$\therefore\ 0 \geq \mathbb{S}_i \geq -1.$$

■

Now, consider the move transformation applied to an arbitrary polygonal fuzzy membership function $A = (a_0, \ldots, a_{n-1})$ to generate $A' = (a'_0, \ldots, a'_{n-1})$ such that they have the same representative value and the same lengths of supports, and $a'_i = a_i + l_i$, $i = \{0, \ldots, \lceil n/2\rceil - 2\}$. In order to achieve this, the move transformation is decomposed to $(\lceil n/2\rceil - 1)$ sub-moves. The ith sub-move $(i = \{0, \ldots, \lceil n/2\rceil - 2\})$ moves the ith support (indexed from the bottom to the top beginning with 0) to its desired place. It moves all the odd points on and above the ith support, whilst keeping unaltered for those points under this support. In particular, in the ith sub-move, the move ratio $\mathbb{M}_i$ is calculated by

$$\mathbb{M}_i = \begin{cases} \dfrac{l_i - \left(a_i^{(i-1)} - a_i\right)}{min\left\{\frac{a_i^{(i-1)} + a_{i+1}^{(i-1)} + \ldots + a_{\lceil \frac{n}{2}\rceil - 1}^{(i-1)}}{\lceil \frac{n}{2}\rceil - i} - a_i^{(i-1)}, a_{n-i}^{(i-1)} - a_{n-1-i}^{(i-1)}\right\}} \\ \left(if\ l_i \geq \left(a_i^{(i-1)} - a_i\right)\right) \\ \dfrac{l_i - \left(a_i^{(i-1)} - a_i\right)}{min\left\{a_{n-1-i}^{i-1} - \frac{a_{n-\lceil \frac{n}{2}\rceil}^{(i-1)} + \ldots + a_{n-1-i}^{(i-1)}}{\lceil \frac{n}{2}\rceil - i}, a_i^{(i-1)} - a_{i-1}^{(i-1)}\right\}} \\ \left(if\ l_i \leq \left(a_i^{(i-1)} - a_i\right)\right) \end{cases} \tag{86}$$

where the notation $a_i^{(i-1)}$ represents a_i's new position after the $(i-1)$th sub-move (not to be confused with the conventional use of such a representation for powers). Initially, $a_i^{(-1)} = a_i$. If $\mathbb{M}_i \in [0,1]$ when $l_i \geq (a_i^{(i-1)} - a_i)$, or $\mathbb{M}_i \in [-1,0]$ when $l_i \leq (a_i^{(i-1)} - a_i)$, the sub-move carried out similarly to (75)–(80) leads to an NCF set $A^{(i)} = \{a_0^{(i)}, \ldots, a_{n-1}^{(i)}\}$ which has the same representative value as A and which has the new point $a_i^{(i)}$ on the desired position, i.e., $\text{Rep}(A^{(i)}) = \text{Rep}(A)$ and $a_i^{(i)} = a_i + l_i$.

Proof: In the ith sub-move ($i = \{0, \ldots, \lceil n/2 \rceil - 2\}$), the odd points under the ith support are not changed:

$$a_j^{(i)} = a_j^{(i-1)}, \qquad j = \{0, \ldots, i-1, n-i, \ldots, n-1\}$$

while the other points $a_i, a_{i+1}, \ldots, a_{n-1-i}$ are being moved. Initially, when $i = 0$, all odd points are being moved of course. If moving to the right direction from the viewpoint of $a_i^{(i-1)}$, i.e., $\mathbb{M}_i \in [0,1]$, the new positions of a_j ($j = \{i, i+1, \ldots, \lceil n/2 \rceil - 1\}$) which are on the left side of fuzzy set A can be computed by

$$a_j^{(i)} = a_j^{(i-1)} + \mathbb{M}_i' \left(\frac{a_i^{(i-1)} + \ldots + a_{\lceil \frac{n}{2} \rceil - 1}^{(i-1)}}{\lceil \frac{n}{2} \rceil - i} - a_j^{(i-1)} \right) \quad (87)$$

where (88), as shown at the bottom of the page, holds. $\mathbb{M}_i'$ is the applied move ratio for the ith sub-move. If $\mathbb{M}_i \in [0,1]$, $\mathbb{M}_i' \in [0, \mathbb{M}_i]$. This avoids the possible below nonconvexity. In the case where $j = i$, substituting (86) and (88) to (87) leads to $a_i^{(i)} = a_i + l_i$, which is the desired position for a_i to be moved to. As the ith support length is fixed, a_{n-1-i} is also moved to the desired position via this sub-move. Initially, the 0th sub-move moves a_0 and a_{n-1} to the correct positions, and the first sub-move moves a_1 and a_{n-2} to the correct positions while keeping a_0 and a_{n-1} unchanged. Following this by induction, the i-th sub-move moves $a_0, \ldots, a_i, a_{n-1-i}, \ldots, a_{n-1}$ to the correct positions.

The distances between $a_{j+1}^{(i)}$ and $a_j^{(i)}$ ($j = \{i, i+1, \ldots, \lceil n/2 \rceil - 2\}$) are calculated as follows, according to (87):

$$a_{j+1}^{(i)} - a_j^{(i)} = \left(a_{j+1}^{(i-1)} - a_j^{(i-1)} \right) (1 - \mathbb{M}_i').$$

Initially, when $i = 0$, $a_{j+1}^{(i-1)} - a_j^{(i-1)} = a_{j+1}^{(-1)} - a_j^{(-1)} = a_{j+1} - a_j \geq 0$ ($j = \{0, 1, \ldots, \lceil n/2 \rceil - 2\}$) as A is convex. This leads to $a_{j+1}^{(0)} - a_j^{(0)} \geq 0$, $j = \{0, 1, \ldots, \lceil n/2 \rceil - 2\}$, which in turn leads to $a_{j+1}^{(1)} - a_j^{(1)} \geq 0$, $j = \{1, \ldots, \lceil n/2 \rceil - 2\}$. Also, as this sub-move causes moves to the right direction, $a_1^{(1)} \geq a_0^{(0)} = a_0^{(1)}$. So $a_{j+1}^{(1)} - a_j^{(1)} \geq 0$, $j = \{0, \ldots, \lceil n/2 \rceil - 2\}$. By induction, it follows that

$$a_{j+1}^{(i)} - a_j^{(i)} \geq 0, \qquad j = \left\{0, \ldots, \left\lceil \frac{n}{2} \right\rceil - 2 \right\}.$$

The new positions of a_j ($j = \{\lfloor n/2 \rfloor, \ldots, n-1-i\}$) which are on the right side of A can be calculated similarly:

$$a_j^{(i)} = a_j^{(i-1)} + \mathbb{M}_i' \left(\frac{a_i^{(i-1)} + \ldots + a_{\lceil \frac{n}{2} \rceil - 1}^{(i-1)}}{\lceil \frac{n}{2} \rceil - i} - a_{n-1-j}^{(i-1)} \right). \quad (89)$$

Thus, the distances between $a_{j+1}^{(i)}$ and $a_j^{(i)}$ ($j = \{\lfloor n/2 \rfloor, \ldots, n-2-i\}$) are calculated by

$$a_{j+1}^{(i)} - a_j^{(i)} = a_{j+1}^{(i-1)} - a_j^{(i-1)} + \mathbb{M}_i' \left(a_{n-1-j}^{(i-1)} - a_{n-2-j}^{(i-1)} \right).$$

Initially, $a_{j+1}^{(-1)} - a_j^{(-1)} + \mathbb{M}_i'(a_{n-1-j}^{(-1)} - a_{n-2-j}^{(-1)}) \geq 0$ ($j = \{\lfloor n/2 \rfloor, \ldots, n-2\}$). This leads to $a_{j+1}^{(0)} - a_j^{(0)} \geq 0$ ($j = \{\lfloor n/2 \rfloor, \ldots, n-2\}$), which in turn leads to $a_{j+1}^{(1)} - a_j^{(1)} \geq 0$ ($j = \{\lfloor n/2 \rfloor, \ldots, n-3\}$). Also, this sub-move ensures $a_{n-1}^{(1)} = a_{n-1}^{(0)} \geq a_{n-2}^{(1)}$, so $a_{j+1}^{(1)} - a_j^{(1)} \geq 0$ ($j = \{\lfloor n/2 \rfloor, \ldots, n-2\}$). Again, by induction

$$a_{j+1}^{(i)} - a_j^{(i)} \geq 0 \quad j = \left\{ \left\lfloor \frac{n}{2} \right\rfloor, \ldots, n-2 \right\}.$$

Thus, it can be summarized that

$$a_{j+1}^{(i)} - a_j^{(i)} \geq 0 \quad j = \{0, \ldots, n-2\}$$

i.e., $A^{(i)}$ is an NCF set.

$$\mathbb{M}_i' = \mathbb{M}_i \frac{min \left\{ \frac{a_i^{(i-1)} + a_{i+1}^{(i-1)} + \ldots + a_{\lceil \frac{n}{2} \rceil - 1}^{(i-1)}}{\lceil \frac{n}{2} \rceil - i} - a_i^{(i-1)}, a_{n-i}^{(i-1)} - a_{n-1-i}^{(i-1)} \right\}}{\frac{a_i^{(i-1)} + a_{i+1}^{(i-1)} + \ldots + a_{\lceil \frac{n}{2} \rceil - 1}^{(i-1)}}{\lceil \frac{n}{2} \rceil - i} - a_i^{(i-1)}} \quad (88)$$

The representative value of A after the ith sub-move, $\mathrm{Rep}(A^{(i)})$, is the same as its original $\mathrm{Rep}(A)$. This is because the following holds according to (87) and (89):

$$\begin{aligned}\mathrm{Rep}\left(A^{(i)}\right) &= \frac{a_0^{(i)} + \ldots + a_{n-1}^{(i)}}{n} \\ &= \frac{a_0^{(i)} + \ldots + a_{n-1}^{(i)}}{n} \\ &= \mathrm{Rep}\left(A^{(i-1)}\right) \\ &= \ldots \\ &= \mathrm{Rep}(A).\end{aligned}$$

The proofs of these properties for moving to the left direction (i.e., $\mathbb{M}_i \in [-1, 0]$) are omitted as they mirror the derivations as given before. ■

In summary, if given move ratios $\mathbb{M}_i \in [-1, 1]$, $(i = \{0, \ldots, \lceil n/2 \rceil - 2\})$, the $(\lceil n/2 \rceil - 1)$ sub-moves transform the given NCF set $A = (a_0, \ldots, a_{n-1})$ to a new NCF set $A' = (a'_0, \ldots, a'_{n-1})$ while keeping their representative values the same.

In the converse case, where two convex fuzzy sets $A = (a_0, \ldots, a_{n-1})$ and $A' = (a'_0, \ldots, a'_{n-1})$ are given which have the same representative value, the move ratio $\mathbb{M}_i$, $i = \{0, 1, \ldots, \lceil n/2 \rceil - 2\}$, is computed by

$$\mathbb{M}_i = \begin{cases} \dfrac{a'_i - a_i^{(i-1)}}{min\left\{\dfrac{a_i^{(i-1)} + \ldots + a_{\lceil \frac{n}{2} \rceil - 1}^{(i-1)}}{\lceil \frac{n}{2} \rceil - i} - a_i^{(i-1)}, a_{n-i}^{(i-1)} - a_{n-1-i}^{(i-1)}\right\}} \\ \left(\text{if } a'_i \geq a_i^{(i-1)}\right) \\ \dfrac{a'_i - a_i^{(i-1)}}{min\left\{a_{n-1-i}^{i-1} - \dfrac{a_{n-\lceil \frac{n}{2} \rceil}^{(i-1)} + \ldots + a_{n-1-i}^{(i-1)}}{\lceil \frac{n}{2} \rceil - i}, a_i^{(i-1)} - a_{i-1}^{(i-1)}\right\}} \\ \left(\text{if } a'_i \leq a_i^{(i-1)}\right) \end{cases} \quad (90)$$

where $a_i^{(i-1)}$ is the a_i's new position after the $(i-1)$th sub-move. Initially, when $i = 0$, $a_i^{(-1)} = a_i$. This sub-move (bottom sub-move) will not lead to underneath nonconvexity as there are no odd points below it, whilst the other sub-moves need to consider situations where nonconvexity arises both above and below. Initially, when $i = 0$, $a_{n-i}^{(i-1)} - a_{n-1-i}^{(i-1)}$, and $a_i^{(i-1)} - a_{i-1}^{(i-1)}$ are not defined. In order to keep integrity of (90), both of them take an infinite value to represent the bottom case.

Given that $A = (a_0, \ldots, a_{n-1})$ and $A' = (a'_0, \ldots, a'_{n-1})$ are both convex, the ranges of $\mathbb{M}_i$ (i.e., $\mathbb{M}_i \in [0, 1]$ when $a'_i \geq a_i^{(i-1)}$ or $\mathbb{M}_i \in [-1, 0]$ when $a'_i \leq a_i^{(i-1)}$) are obvious and hence no proof is needed.

Note that this work is readily extendable to rules involving variables that are represented by Gaussian and other bell-shaped membership functions. For instance, consider the simplest case where two rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ and the observation A^* all involve the use of Gaussian fuzzy sets of the form (Fig. 14)

$$p(x) = e^{\frac{-(x-c)^2}{2\sigma^2}} \quad (91)$$

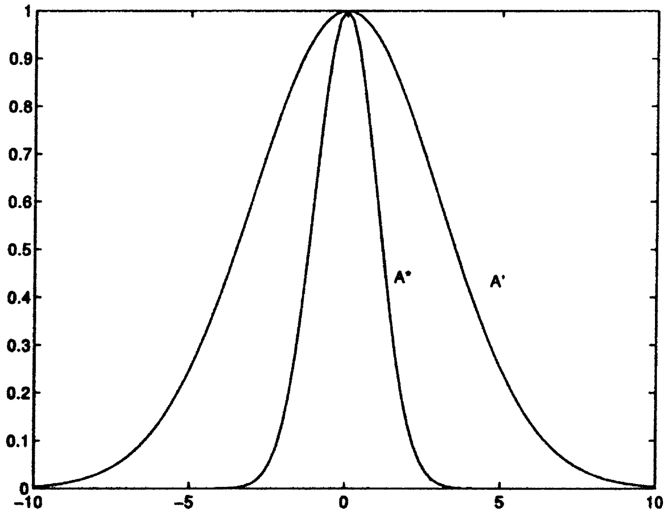

Fig. 14. Gaussian scale transformation.

where c and σ are the mean and standard deviation, respectively. The construction of the intermediate rule is slightly different from the polygonal fuzzy membership function cases in the sense that the standard deviations are used to interpolate. Since the Gaussian shape is symmetrical, c is chosen to be the representative value of such a fuzzy set. In so doing, the antecedent value A' of the intermediate rule has the same representative value as that of observation A^*. That means only scale transformation from A' to A^* as depicted in Fig. 14 is needed to carry out interpolation. Heuristics can be employed to represent the scale rate s in terms of the standard deviation σ. One of the simplest definitions is to calculate the ratio of two fuzzy sets' σ values when considering transformation from one to the other. The scale rate s can, therefore, be written as

$$s = \frac{\sigma_{A^*}}{\sigma_{A'}}. \quad (92)$$

The transformations involving other bell-shaped membership functions follows this idea analogously.

E. Integrated Transformation and the Summary of Interpolation Procedure

On top of the *scale* and *move transformations*, an *integrated transformation*, denoted as $T(A, A')$, between two arbitrary fuzzy sets $A = (a_0, \ldots, a_{n-1})$ and $A' = (a_0, \ldots, a_{n-1})$ can be introduced such that A' is the derived NCF set of A by applying both transformation components. An integrated transformation includes: 1) the information of a scale rate and one or more scale ratios used in scale transformation, depending upon whether triangular or more complex polygonal fuzzy membership functions are dealt with, and 2) the information of one or more move ratios in move transformation, again depending upon whether triangular, trapezoidal or more complex polygonal fuzzy membership functions are used. In general, an integrated transformation can be represented as

$$T(A, A') = \left\{s_0, \mathbb{S}_1, \mathbb{S}_2, \ldots, \mathbb{S}_{\lfloor \frac{n}{2} \rfloor - 1}, \mathbb{M}_0, \mathbb{M}_1, \ldots, \mathbb{M}_{\lceil \frac{n}{2} \rceil - 2}\right\} \quad (93)$$

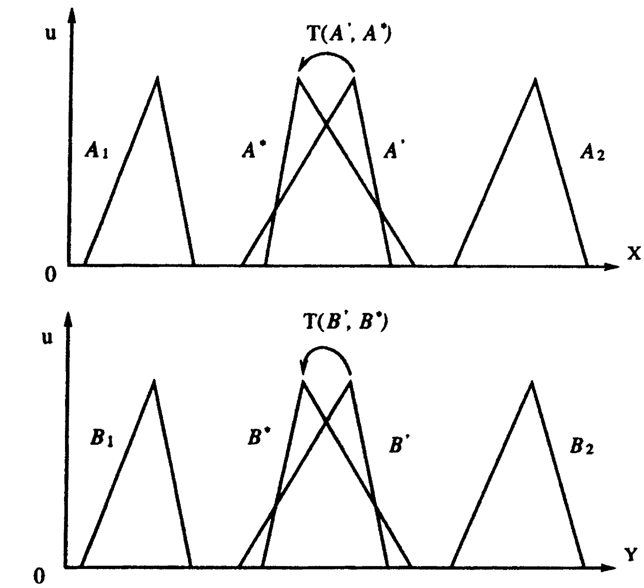

Fig. 15. Proposed interpolative method.

Obviously, the integrated transformation $T(A, A')$ and another, say

$$T(B,B')=\left\{s'_0, \mathbb{S}'_1, \mathbb{S}'_2, \ldots, \mathbb{S}'_{\lfloor \frac{n}{2} \rfloor -1}, \mathbb{M}'_0, \mathbb{M}'_1, \ldots, \mathbb{M}'_{\lceil \frac{n}{2} \rceil -2}\right\}$$

are said to be identical if and only if both of their corresponding *scale rate*, *scale ratios*, and *move ratios* are equal, respectively

$$\begin{cases} s_0 = s'_0 \\ \mathbb{S}_i = \mathbb{S}'_i \\ \mathbb{M}_j = \mathbb{M}'_j \end{cases} \tag{94}$$

where $i = \{1, \ldots, \lfloor n/2 \rfloor - 1\}$ and $j = \{0, \ldots, \lceil n/2 \rceil - 2\}$.

As indicated earlier, it is intuitive to maintain the similarity degree between the consequent parts $B' = (b'_0, \ldots, b'_{n-1})$ and $B^* = (b^*_0, \ldots, b^*_{n-1})$ to be the same as that between the antecedent parts $A' = (a'_0, \ldots, a'_{n-1})$ and $A^* = (a^*_0, \ldots, a^*_{n-1})$, in performing interpolative reasoning. Now, that the integrated transformation allows the similarity degree between two fuzzy sets to be measured by the *scale rate*, *scale ratios*, and *move ratios*, the desired conclusion B^* can be obtained by satisfying the following (as shown in Fig. 15 for an interpolation involving triangular fuzzy sets):

$$T(B', B^*) = T(A', A^*). \tag{95}$$

That is, the *scale rate*, *scale ratios*, and *move ratios* calculated from A' to A^* are used to compute B^* from B'. The computation procedure is summarized as follows.

1) Calculate scale rates s_i $(i = \{0, 1 \ldots, \lfloor n/2 \rfloor - 1\})$ of the ith support from A' to A^* as follows:

$$s_i = \frac{a^*_{n-1-i} - a^*_i}{a'_{n-1-i} - a'_i}. \tag{96}$$

2) Calculate scale rate s_0 of the bottom support and scale ratios $\mathbb{S}_i$ $(i = \{1 \ldots, \lfloor n/2 \rfloor - 1\})$ of the ith support from A' to A^* according to (85)

$$s_0 = \frac{a^*_{n-1} - a^*_0}{a'_{n-1} - a'_0} \tag{97}$$

$$\mathbb{S}_i = \begin{cases} \dfrac{\frac{a^*_{n-i-1}-a^*_i}{a^*_{n-i}-a^*_{i-1}} - \frac{a'_{n-i-1}-a'_i}{a'_{n-i}-a'_{i-1}}}{1-\frac{a'_{n-i-1}-a'_i}{a'_{n-i}-a'_{i-1}}} \\ \left(\text{if } \frac{a^*_{n-i-1}-a^*_i}{a^*_{n-i}-a^*_{i-1}} \geq \frac{a'_{n-i-1}-a'_i}{a'_{n-i}-a'_{i-1}} \geq 0\right) \\ \dfrac{\frac{a^*_{n-i-1}-a^*_i}{a^*_{n-i}-a^*_{i-1}} - \frac{a'_{n-i-1}-a'_i}{a'_{n-i}-a'_{i-1}}}{\frac{a'_{n-i-1}-a'_i}{a'_{n-i}-a'_{i-1}}} \\ \left(\text{if } \frac{a'_{n-i-1}-a'_i}{a'_{n-i}-a'_{i-1}} \geq \frac{a^*_{n-i-1}-a^*_i}{a^*_{n-i}-a^*_{i-1}} \geq 0\right). \end{cases} \tag{98}$$

As A' and A^* are both convex, $\mathbb{S}_i \in [0, 1]$ (when $s_i \geq s_{i-1}$) or $\mathbb{S}_i \in [-1, 0]$ (when $s_{i-1} \geq s_i$) must hold according to the proof given in III-D.

3) Apply scale transformation to A' with scale rates s_i $(i = \{0, 1 \ldots, \lfloor n/2 \rfloor - 1\})$ calculated in the first step to obtain A'' by simultaneously solving n linear equations. As $\mathbb{S}_i \in [-1, 1]$ $(i = \{0, 1 \ldots, \lfloor n/2 \rfloor - 1\})$, it enables A'' to have all its support lengths arranged in descending order from the bottom to the top. This, together with the scale transformation, guarantees A'' to be a unique, normal and convex fuzzy set, which has the same representative value as A^* and has the same $\lfloor n/2 \rfloor$ support lengths as those of A^*.

4) Assign scale rate s'_0 of the bottom support of B' to the value of s_0 (i.e., $s'_0 = s_0$) as it does not give rise to nonconvexity. The scale ratios $\mathbb{S}'_i$, $(i = \{1 \ldots, \lfloor n/2 \rfloor - 1\})$ of the ith support of B' are in the form

$$\mathbb{S}'_i = \begin{cases} \dfrac{\frac{s'_i\left(b^*_{n-i-1}-b^*_i\right)}{s'_{i-1}\left(b^*_{n-i}-b^*_{i-1}\right)} - \frac{b^*_{n-i-1}-b^*_i}{b^*_{n-i}-b^*_{i-1}}}{1-\frac{b^*_{n-i-1}-b^*_i}{b^*_{n-i}-b^*_{i-1}}} & if\ s_i \geq s_{i-1} \geq 0 \\ \dfrac{\frac{s'_i\left(b^*_{n-i-1}-b^*_i\right)}{s'_{i-1}\left(b^*_{n-i}-b^*_{i-1}\right)} - \frac{b^*_{n-i-1}-b^*_i}{b^*_{n-i}-b^*_{i-1}}}{\frac{b^*_{n-i-1}-b^*_i}{b^*_{n-i}-b^*_{i-1}}} & if\ s_{i-1} \geq s_i \geq 0. \end{cases} \tag{99}$$

They are required to equal $\mathbb{S}_i$ $(i = \{1 \ldots, \lfloor n/2 \rfloor - 1\})$ in step 2). Solving this along with the initial status $(s'_0 = s_0)$ leads to the following scale rates s'_i $(i = \{0, 1 \ldots, \lfloor n/2 \rfloor - 1\})$:

$$s'_i = \begin{cases} s_i, & if\ i=0 \\ \dfrac{s'_{i-1}(s_i - s_{i-1})\left(\frac{b'_{n-i}-b'_{i-1}}{b'_{n-i-1}-b'_i}-1\right)}{s_{i-1}\left(\frac{a'_{n-i}-a'_{i-1}}{a'_{n-i-1}-a'_i}-1\right)} + s'_{i-1}, & if\ s_i \geq s_{i-1} \geq 0 \\ \dfrac{s'_{i-1}s_i}{s_{i-1}}, & if\ s_{i-1} \geq s_i \geq 0. \end{cases} \tag{100}$$

5) Apply scale transformation to B' with s'_i $(i = \{0, 1 \ldots, \lfloor n/2 \rfloor - 1\})$ calculated in step 4) to obtain $B'' = (b''_0, \ldots, b''_{n-1})$, by simultaneously solving the n linear equations. As B' is convex and $\mathbb{S}'_i = \mathbb{S}_i \in [-1, 1]$, it enables B'' to have descending support lengths from

the bottom to the top. This, together with the scale transformation, ensures B'' to be a unique, normal and convex fuzzy set.

6) Decompose the move transformation to $(\lceil n/2 \rceil - 1)$ sub-moves. For $i = 0, 1, \ldots, \lceil n/2 \rceil - 2$, carry out the following.
   a) Calculate move ratio $\mathbb{M}_i$ of the ith support from A'' according to (90)

$$\mathbb{M}_i = \begin{cases} \dfrac{a_i^* - a_i^{(i-1)}}{min\left\{\dfrac{a_i^{(i-1)}+\ldots+a_{\lceil \frac{n}{2} \rceil -1}^{(i-1)}}{\lceil \frac{n}{2} \rceil - i} - a_i^{(i-1)}, a_{n-i}^{(i-1)} - a_{n-1-i}^{(i-1)}\right\}} \\ \left(\text{if } a_i^* \geq a_i^{(i-1)}\right) \\ \dfrac{a_i^* - a_i^{(i-1)}}{min\left\{a_{n-1-i}^{(i-1)} - \dfrac{a_{n-\lceil \frac{n}{2} \rceil}^{(i-1)}+\ldots+a_{n-1-i}^{(i-1)}}{\lceil \frac{n}{2} \rceil - i}, a_i^{(i-1)} - a_{i-1}^{(i-1)}\right\}} \\ \left(if\ a_i^* \leq a_i^{(i-1)}\right) \end{cases} \tag{101}$$

   where $a_i^{(i-1)}$ is a_i' new position after $(i-1)$ sub-moves. Initially, when $i = 0$, $a_i^{(-1)} = a_i''$ $(i = \{0, \ldots, n-1\})$. As $A^{(i-1)}$ and A^* are both convex, $\mathbb{M}_i \in [-1, 1]$.
   b) Apply move transformation to $A^{(i-1)}$ with $\mathbb{M}_i$ to obtain $A^{(i)} = \{a_0^{(i)}, a_1^{(i)}, \ldots, a_n^{(i)}\}$. As $\mathbb{M}_i \in [-1, 1]$ and $A^{(i-1)}$ is convex, $A^{(i)}$ must be convex according to the proof given in III-D.
   c) Apply move transformation to $B^{(i-1)}$ with $\mathbb{M}_i$ to obtain $B^{(i)} = \{b_0^{(i)}, b_1^{(i)}, \ldots, b_n^{(i)}\}$. Again, it must be convex.
7) When the *for* loop of step 6) is terminated, the procedure returns that $A^{(\lceil n/2 \rceil - 2)} = A^*$ and $B^{(\lceil n/2 \rceil - 2)}$, which is the resultant fuzzy set B^*.

Clearly, B' and B^* will then retain the same similarity degree as that between the antecedent parts A' and A^*.

Note that the summarized procedure above implicitly assumes that all fuzzy sets involved in the interpolation have the same number of the odd points. However, this is not always the case. Fortunately, this can be easily extended by assigning all fuzzy sets the same odd point number, which is set to the number of the odd points of the fuzzy set that has the most such points among all the fuzzy sets involved.

There are two specific cases worth noting when applying the scale transformation. The first is that if A^* is a singleton while A' is a regular normal and convex fuzzy set, the scale transformation from A' to A^* is 0. This case can be handled by setting the result B^* to a singleton whose value interpolates between $\text{Rep}(B_1)$ and $\text{Rep}(B_2)$ in the same way as A^* does between $\text{Rep}(A_1)$ and $\text{Rep}(A_2)$. The second case (which only exists if both antecedents A_1 and A_2 are singletons) is that if A^* is a regular normal and convex fuzzy set while A' is a singleton, the scale transformation from A' to A^* would be infinite. Since infinity cannot be used to generate the resulting fuzzy set, a modified method is created for this. The ratio between the support length of A^* and the distance from $\text{Rep}(A_1)$ to $\text{Rep}(A_2)$ is calculated. It is used to equalize the ratio between the support length of B^* and the distance from $\text{Rep}(B_1)$ to $\text{Rep}(B_2)$, by which the support length of B^* can be obtained. Note that the fuzzy set obtained by the scale transformation from a singleton is an isosceles polygon.

It is desirable for any fuzzy interpolation technique to give prompt response when it is used to handle time critical applications. Therefore, complexity analysis in terms of time and space is an important issue for the present method as well as for others. Most attention is drawn to the analysis of time complexity as the space problem nearly vanishes due to the rapid development of the storage capacity and the method's embedded power of being able to handle sparse rule bases. With respect to n (the largest number of odd points for any fuzzy sets involved), the transformation-based interpolation needs $O(n^2)$ computation time mainly owing to step 6) in the calculation of the interpolative results. This is acceptable given that n is not significantly large in most cases, and that high-speed processors are more and more popularly used.

F. Multiple Antecedent Variables Interpolation

The one variable case described above concerns interpolation between two adjacent rules with each involving one antecedent variable. This is readily extendable to rules with multiple antecedent attributes. Of course, the attributes appearing in both rules must be the same to make sense for interpolation.

Without losing generality, suppose that two adjacent rules R_i and R_j are represented by

$$\text{if } X_1\ is\ A_{1i} \text{ and} \ldots \text{and } X_m\ is\ A_{mi} \text{ then } Y \text{ is } B_i,$$
$$\text{if } X_1 \text{ is } A_{1j} \text{ and} \ldots \text{and } X_m \text{ is } A_{mj} \text{ then } Y \text{ is } B_j.$$

Thus, when a vector of observations $((A_1^*, \ldots, A_k^*, \ldots, A_m^*)$ is given, by direct analogy to one variable case, the values A_{ki} and A_{kj} of X_k, $k = 1, 2, \ldots, m$, are used to obtain the new NCF set A'_k

$$A'_k = (1 - \lambda_k)A_{ki} + \lambda_k A_{kj} \tag{102}$$

where

$$\lambda_k = \frac{d(\text{Rep}(A_{ki}), \text{Rep}(A_k^*))}{d(\text{Rep}(A_{ki}), \text{Rep}(A_{kj}))}.$$

Clearly, the representative value of A'_k will remain the same as that of the k-th observation A_k^*.

The resulting A'_k and the given A_k^* are used to compute the integrated transformation

$$T(A'_k, A_k^*) = \Big\{s_{k0}, \mathbb{S}_{k1}, \ldots, \mathbb{S}_{k(\lfloor \frac{n}{2} \rfloor - 1)}, \mathbb{M}_{k0}, \ldots, \mathbb{M}_{k(\lceil \frac{n}{2} \rceil - 2)}\Big\}$$

just like the one variable case. From this, the combined scale rate s_c, scale ratios $\mathbb{S}_{ci}$, $(i = \{1, \ldots, \lfloor n/2 \rfloor - 1\})$ and move ratios $\mathbb{M}_{cj}$ $(j = \{0, \ldots, \lceil n/2 \rceil - 2\})$ over the m conditional attributes are, respectively, calculated as the arithmetic averages of s_{k0}, $\mathbb{S}_{ki}$ and $\mathbb{M}_{kj}$, $k = 1, 2, \ldots, m$

$$s_c = \frac{1}{m} \sum_{k=1}^{m} s_{k0} \tag{103}$$

$$\mathbb{S}_{ci} = \frac{1}{m}\sum_{k=1}^{m}\mathbb{S}_{ki} \quad (104)$$

$$\mathbb{M}_{cj} = \frac{1}{m}\sum_{k=1}^{m}\mathbb{M}_{kj}. \quad (105)$$

Note that, other than using arithmetic average, different mechanisms such as the medium value operator may be employed for this purpose. However, the average helps to capture the intuition that when no particular information regarding which variable has a more dominating influence upon the conclusion, all the variables are treated equally. If such information is available, a weighted average operator may be better to use.

Regarding the consequent, by analogy to expression (22), B' can be computed by

$$B' = (1-\lambda_a)B_i + \lambda_a B_j. \quad (106)$$

Here, λ_a is deemed to be the average of λ_k, $k = 1, 2, \ldots, m$, to mirror the approach taken previously

$$\lambda_a = \frac{1}{m}\sum_{k=1}^{m}\lambda_k. \quad (107)$$

As the integrated transformation

$$T = \left\{s_c, \mathbb{S}_{c1}, \mathbb{S}_{c2}, \ldots, \mathbb{S}_{c\lfloor \frac{n}{2}\rfloor -1}, \mathbb{M}_{c0}, \mathbb{M}_{c1}, \ldots, \mathbb{M}_{c\lceil \frac{n}{2}\rceil -2}\right\}$$

reflects the similarity degree between the observation vector and the values of the antecedent variables in the given rules, the fuzzy set B^* of the conclusion can then be estimated by transforming B' via the application of the same T.

TABLE II
RESULTS FOR EXAMPLE 1, WITH $A^* = (7, 8, 9)$

Attribute Values	Results	
$A_1 = (0, 5, 6)$	Method	B^*
$A_2 = (11, 13, 14)$	KH	(6.36, 5.38, 7.38)
$B_1 = (0, 2, 4)$	HCL	(6.36, 6.58, 7.38)
$B_2 = (10, 11, 13)$	HS	(5.83, 6.26, 7.38)

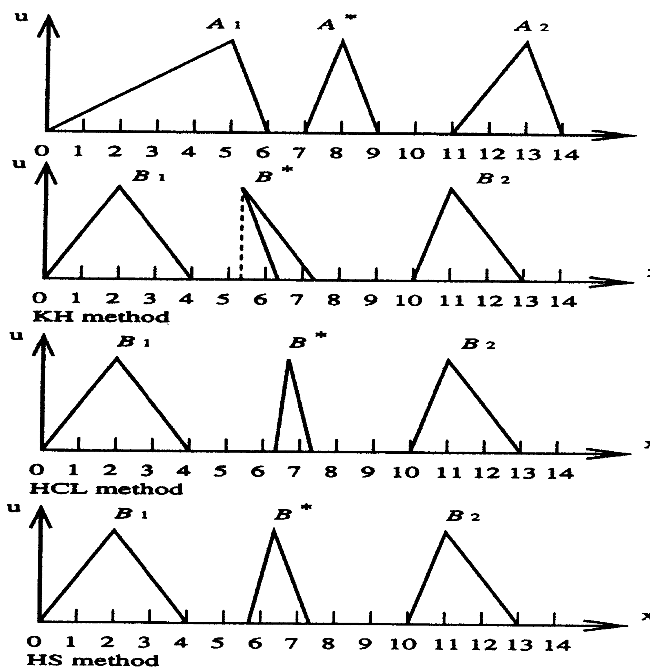

Fig. 16. Example 1.

IV. EXPERIMENTAL RESULTS

In this section, the example problems given in [4] and [17], together with several new problem cases are used to illustrate the newly proposed interpolative method and to facilitate comparative studies. All the results except Example 7 discussed below concern the interpolation between two adjacent rules $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$, while Example 7 shows a case of interpolation between rules involving two antecedent variables. In reporting these results, HCL stands for the work of [4] and HS stands for the work proposed in this paper (and KH stands for the method given in [7] and [8], as stated before).

Example 1: This example demonstrates the use of the proposed method involving only triangular fuzzy sets. All the conditions are shown in Table II and Fig. 16, which also include the results of interpolation. Suppose $A^* = (7, 8, 9)$. First, according to (18) and (22), $A'(5.30, 8.85, 9.85)$ and $B'(4.81, 6.33, 8.33)$ are calculated by interpolation of A_1, A_2 and B_1, B_2, respectively, with $\lambda_{\mathrm{Rep}} = 0.48$, which is calculated from (14). Then, the scale rate $s = 0.44$ and the move rate $m = 0.36$ in the integrated transformation from A' and A^* are calculated with regard to (28) and (34). Finally, the s and m are used to transform B' according to (25)–(27) and (29)–(31), resulting in consequence $B^*(5.83, 6.26, 7.83)$. For this case, the KH method resulted in a nonconvex conclusion while the other two concluded with normal and convex fuzzy sets. Note that the consequent obtained by the KH method is not even a membership function.

Example 2: The second case considers when the scale rate is infinity. The given observation is a triangular fuzzy set (5,6,8). Table III and Fig. 17 present the antecedents and interpolated fuzzy sets. The result of the interpolation $B = (5.71, 6.28, 8.16)$ is obtained as follows: First, the ratio between the support of A^* and the distance of $\mathrm{Rep}(A_1)$ and $\mathrm{Rep}(A_2)$ is calculated, then the support of B^* is computed by retaining the same ratio but based on the distance of $\mathrm{Rep}(B_1)$ and $\mathrm{Rep}(B_2)$, and finally, the move transformation is applied as usual. The comparative results show that the KH and HCL methods performed similarly (the supports of the resultant fuzzy sets are identical since they are computed in the same way) while the HS method also generated a very reasonable outcome.

Example 3: The third case considers a similar situation to example 1 but the observation is a singleton $A^* = (8, 8, 8)$. Table IV and Fig. 18 present the results. In this case, the KH method once again generated a nonconvex fuzzy set and the HCL method even produced a nontriangular fuzzy set. However, the method proposed in this paper resulted in a singleton conclusion, which is rather intuitive given the singleton-valued condition.

TABLE III
RESULTS FOR EXAMPLE 2, WITH $A^* = (5, 6, 8)$

Attribute Values	Results	
$A_1 = (3, 3, 3)$	Method	B^*
$A_2 = (12, 12, 12)$	KH	(5.33, 6.33, 9.00)
$B_1 = (4, 4, 4)$	HCL	(5.33, 6.55, 9.00)
$B_2 = (10, 11, 13)$	HS	(5.71, 6.28, 8.16)

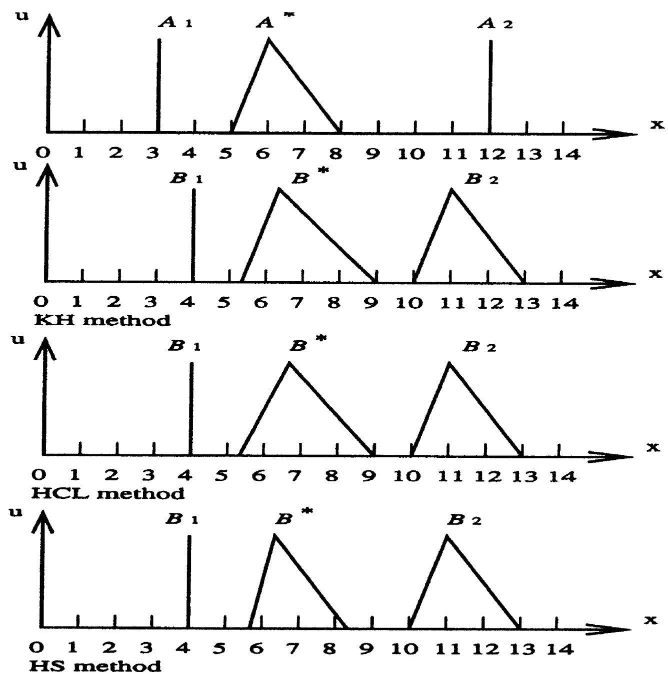

Fig. 17. Example 2.

TABLE IV
RESULTS FOR EXAMPLE 3, WITH $A^* = (8, 8, 8)$

Attribute Values	Results	
$A_1 = (0, 5, 6)$	Method	B^*
$A_2 = (11, 13, 14)$	KH	(7.27, 5.38, 6.25)
$B_1 = (0, 2, 4)$	HCL	[7.27,6.25]
$B_2 = (10, 11, 13)$	HS	(6.49, 6.49, 6.49)

Example 4: This example concerns a trapezoidal interpolation. As there is no obvious indication for HCL method to handle trapezoidal fuzzy sets, only KH method is used in comparison. All the attributes and results with observation $A^* = (6, 6, 9, 10)$ are shown in Table V and Fig. 19. First, $A' = (5.30, 7.85, 8.85.9.85)$ and $B' = (4.81, 6.33, 7.33, 8.33)$ are calculated by interpolation of A_1, A_2 and B_1, B_2, respectively, with $\lambda = 0.48$, which is calculated from (14). Then, the interpolation via scale and move transformations is carried out according to the steps listed in Section III-E: 1) The bottom support scale rate (0.88) and top support scale rate (3.0) from A' to A^* are calculated, respectively, according to (96). 2) The top support scale ratio (0.68) from A' to A^* is calculated with respect to (98). 3) A' is scaled to generate $A'' = (5.76, 6.48, 9.48, 9.76)$ using the bottom and top scale rates calculated in step 1). Note that A'' is a convex fuzzy set which has the same representative value and has the same bottom and top support lengths as A^*. 4) According to (100), the bottom and top support scale rates (0.88 and 2.38) over B' are computed. 5) B' is scaled to generate $B'' = (5.09, 5.52, 7.90, 8.18)$ using the bottom and top scale rates calculated in step 4). 6) The move ratio is calculated from

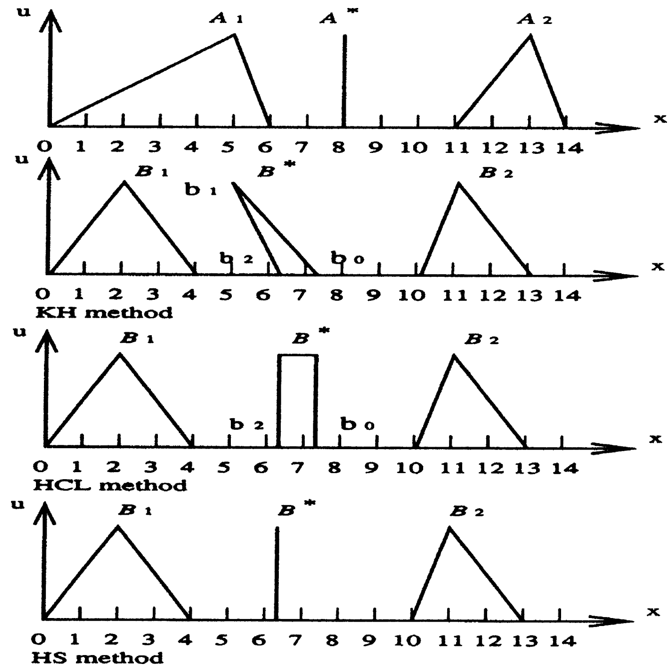

Fig. 18. Example 3.

TABLE V
RESULTS FOR EXAMPLE 4, WITH $A^* = (6, 6, 9, 10)$

Attribute Values	Results	
$A_1 = (0, 4, 5, 6)$	Method	B^*
$A_2 = (11, 12, 13, 14)$	KH	(5.45, 4.25, 7.5, 8.5)
$B_1 = (0, 2, 3, 4)$	HCL	-
$B_2 = (10, 11, 12, 13)$	HS	(5.23, 5.23 ,7.61, 8.32)

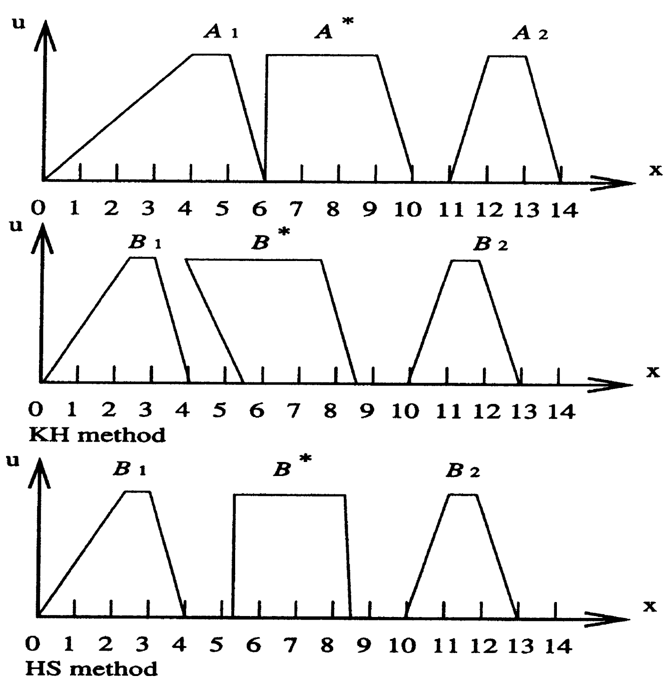

Fig. 19. Example 4.

TABLE VI
RESULTS FOR EXAMPLE 5, WITH $A^* = (6, 6.5, 7, 9, 10, 10.5)$

Attribute Values	Results	
$A_1 = (0, 1, 3, 4, 5, 5.5)$	Method	B^*
	KH	(5.73, 6.00,5.89, 8.56, 9.59, 10.09)
$A_2 = (11, 11.5, 12, 13, 13.5, 14)$	HCL	-
	HS1	(5.64, 5.98, 6.29, 8.63, 9.46, 9.93)
$B_1 = (0, 0.5, 1, 3, 4, 4.5)$	HS2	(5.69, 6.03, 6.36, 8.69, 9.53, 10.00)
	HS3	(5.61, 5.95, 6.26, 8.59, 9.42, 9.89)
$B_2 = (10.5, 11, 12, 13, 13.5, 14)$	HS4	(5.47, 5.79, 6.08, 8.42, 9.23, 9.70)

A'' to A^* according to (101). Its value is 1.0 as A^* has left vertical slope. This move ratio is used to move B'' to obtain the resultant fuzzy set $B^* = (5.23, 5.23, 7.61, 8.32)$. In this case, the KH method once again generated a nonconvex fuzzy set (which does not satisfy the definition of a membership function). However, the HS method resulted in a convex conclusion, which still has a left vertical slope.

Example 5: This example shows an interpolation of rules involving hexagonal fuzzy sets interpolation, and it also demonstrates the flexibility of the proposed method to generate multiple results via different representative value definitions (including average, compatible, weighted average and center of core). As all the processes follow in the same way, only the interpolation using the average representative value is described in details. Again, there is no obvious indication for HCL method to handle such fuzzy sets, only KH method is used in comparison.

All the attribute values and results with respect to the observation $A^* = (6, 6.5, 7, 9, 10, 10.5)$ are shown in Table VI and Fig. 20. HS1, HS2, HS3, and HS4 indicate the HS interpolation using average, compatible, weighted average and center of core representative values respectively. Note that in this example, the two intermediate points a_1 and a_4 of each fuzzy set involved have a membership value of 0.5. First, with respect to the *average definition*, $A' = (5.94, 6.67, 7.86, 8.86, 9.59, 10.09)$ and $B' = (5.67, 6.17, 6.94, 8.40, 9.13, 9.63)$ are calculated by interpolation of A_1, A_2 and B_1, B_2 (with $\lambda = 0.54$), respectively. Then, the interpolation via scale and move transformations is carried out according to the steps listed in Section III-E: 1) The bottom support scale rate (1.08), middle support scale rate (1.20) and top support scale rate (2.0) from A' to A^* are calculated according to (96), respectively. 2) The middle and top support scale ratios (0.25 and 0.35) from A' to A^* are calculated with respect to (98). 3) A' is scaled to generate $A'' = (5.79, 6.39, 7.32, 9.32, 9.89, 10.29)$ using the bottom, middle and top scale rates calculated in step 1). Note that A'' is a convex fuzzy set which has the same representative value and the same three support lengths as A^*. 4) According to (100), the bottom, middle, and top support scale rates (1.08, 1.18, and 1.60) over B' are computed. 5) B' is scaled to generate $B'' = (5.50, 5.91, 6.50, 8.83, 9.39, 9.80)$ using the scale rates calculated in step 4). 6) Two sub-moves are required in performing the move transformation in this case: 6.1), The bottom sub-move ratio (0.29) is calculated from A'' to A^* according to (101). This sub-move ratio is used to move A'' to get $A^{(0)} = (6.00, 6.42, 7.08, 9.08, 9.92, 10.50)$, and to move B'' to obtain $B^{(0)} = (5.64, 5.93, 6.35, 8.68, 9.41, 9.93)$. Note that after this sub-move, A'' has the same bottom support as A^*.

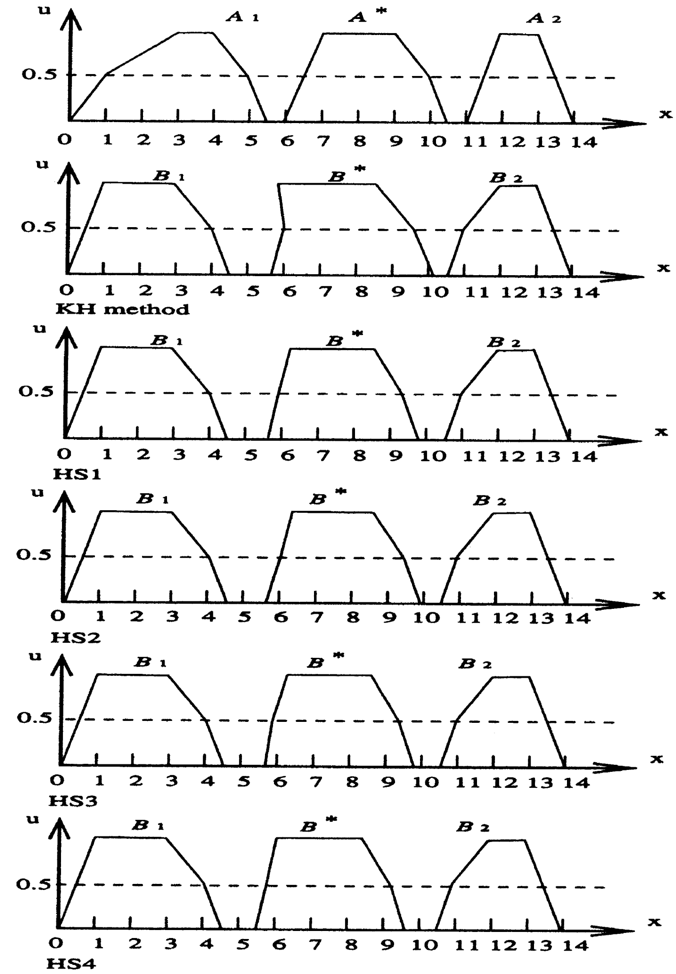

Fig. 20. Example 5.

6.2) The second sub-move moves the middle and top supports of $A^{(0)}$ to the desired places. In particular, the sub-move ratio (0.24) calculated from (101) is used to move $B^{(0)}$ to the final result $B^* = (5.64, 5.98, 6.29, 8.63, 9.46, 9.93)$. As a verification, A^* is obtained by moving $A^{(0)}$ with the same sub-move ratio.

In this case, the HS methods still ensure unique, normal and convex fuzzy sets, compared to the nonconvex result generated via the KH method. It is interesting to note that the four HS results have almost the same geometrical shape although their positions are slightly different. This is because all the calculations involved are the same except the computation of the representative values. This empirically shows that although different representative values may be chosen for use given a specific problem, their influence on the final interpolative outcomes is not significant. This helps ensure the stability of the inference method developed.

Example 6: This case considers an interpolation with Gaussian membership functions. As there are no explicitly Gaussian based interpolation solutions for HCL and KH methods, only the results of HS method together with the attribute values and observation $A^* = p(x) = e^{(-(x-8)^2)/(2*1^2)}$ are presented in Table VII and Fig. 21. Application of the HS method results in a sensible Gaussian conclusion in this case.

TABLE VII
RESULTS FOR EXAMPLE 6, WITH $A^* = e^{(-(x-8)^2)/(2*1^2)}$

Attribute Values	Results	
$A_1 = e^{\frac{-(x-3)^2}{2*2^2}}$	Method	B^*
$A_2 = e^{\frac{-(x-11)^2}{2*0.5^2}}$	KH	-
$B_1 = e^{\frac{-(x-6)^2}{2*1^2}}$	HCL	-
$B_2 = e^{\frac{-(x-13)^2}{2*1.5^2}}$	HS	$e^{\frac{-(x-10.38)^2}{2*1.24^2}}$

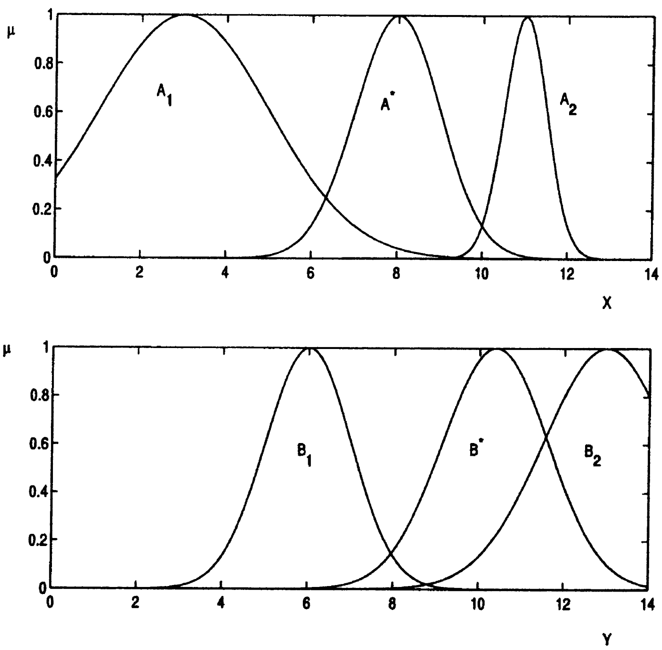

Fig. 21. Example 6.

TABLE VIII
RESULTS FOR EXAMPLE 7, WITH $A_1^* = (6, 7, 9, 11)$ AND $A_2^* = (6, 8, 10, 12)$

Attribute Values	Results	
$A_{11} = (0, 4, 5, 6)$	Method	B^*
$A_{12} = (12, 14, 15, 16)$	KH	(5.45, 5.94, 7.13, 8.31)
$A_{21} = (11, 12, 13, 14)$	HCL	-
$A_{22} = (1, 2, 3, 4)$	HS	(4.37, 5.55, 7.48, 9.33)
$B_1 = (0, 2, 3, 4)$		
$B_2 = (10, 11, 12, 13)$		

Example 7: This example concerns an interpolation of multiple antecedent variables with trapezoidal membership functions. In particular, two rules $A_{11} \wedge A_{21} \Rightarrow B_1$, $A_{12} \wedge A_{22} \Rightarrow B_2$ and the observations $A_1^* = (6, 7, 9, 11)$, $A_2^* = (6, 8, 10, 12)$ are given to determine the result B^*. Table VIII and Fig. 22 summarize the results. In this case, the parameters λ_1 for the first variable is 0.54 and λ_2 for the second is 0.44, the average 0.49 is used to calculate the intermediate rule result B'. Then the average of two bottom support scale rates (1.14 and 1.69) and the average of two top support ratios (0.22 and 0.07) are computed, equalling 1.41 and 0.15 respectively, and are used as the combined bottom support scale rate and top support scale ratio. These, together with the combined move rate, namely the average (0.35) of the two move rates (0.53 and 0.18), are employed to transfer B' to achieve the final result B^*. Both the KH method

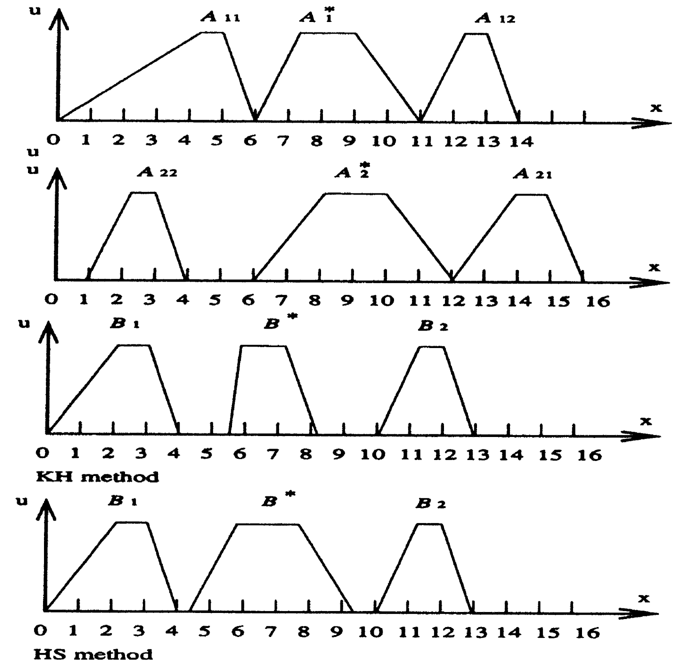

Fig. 22. Example 7.

and the HS method resulted in a convex set in this example. Interestingly, the resultant fuzzy set of the present work reflects better in terms of the shapes of the original observations than that obtained by the KH method. More investigations into the underlying reasons for this are currently being carried out.

V. CONCLUSION

This paper has proposed a generalized, scale and move transformation-based, interpolative reasoning method which can handle interpolation of complex polygon, Gaussian and other bell-shaped fuzzy membership functions. The method works by first constructing a new intermediate rule via manipulating two adjacent rules (and the given observations of course), and then converting the intermediate inference result into the final, derived conclusion, using the scale and move transformations. This approach not only inherits the common advantages of fuzzy interpolative reasoning—allowing inferences to be performed with simple and sparse rule bases, but also has other three advantages. 1) It can handle interpolation of multiple antecedent variables with simple computation. 2) It guarantees the uniqueness as well as normality and convexity of the resulting interpolated fuzzy sets. 3) It suggests a variety of definitions for representative values, providing a degree of freedom in fuzzy modeling to meet different requirements.

There is still room to improve the present work. In particular, as indicated earlier, any underlying theoretical reasons why this work performs better than the KH method in the interpolation of rules involving multiple antecedent variables needs further investigation. In addition, the present work only uses two rules to conduct interpolation, but interpolation involving more rules may be utilized in fuzzy modeling. An extension of the proposed method to cope with such a problem is desirable. Finally, this work does not look into the extrapolation problem. Further effort to investigate this issue seems useful.

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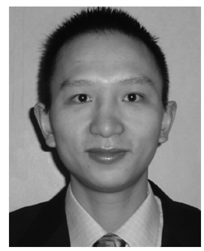

Zhiheng Huang received the B.Sc. degree in industrial equipment and control engineering and computer science from South China University of Technology, P. R. China, and the M.Sc. degree in artificial intelligence from the University of Edinburgh, U.K., in 2000 and 2001, respectively. He is currently working toward the Ph.D. degree in the School of Informatics at the University of Edinburgh, working in Centre for Intelligent Systems and their Applications.

His research interests include fuzzy set theory, fuzzy interpolation, machine learning, and rule-based model simplification.

Qiang Shen received the B.Sc. and M.Sc. degrees in communications and electronic engineering from the National University of Defence Technology, China, and the Ph.D. degree in knowledge-based systems from Heriot-Watt University, Edinburgh, U.K.

He is a Professor with the Department of Computer Science at the University of Wales, Aberystwyth, U.K. His research interests include fuzzy and imprecise modeling, model-based inference, pattern recognition, and knowledge refinement and reuse. He has published 160 peer-refereed papers in academic journals and conferences on topics within artificial intelligence and related areas.

Dr. Shen is an Associate Editor of the IEEE TRANSACTIONS ON FUZZY SYSTEMS, an Associate Editor of the IEEE TRANSACTIONS ON SYSTEMS, MAN, AND CYBERNETICS, PART B: CYBERNETICS, and an Editorial Board member of the journal *Fuzzy Sets and Systems*.